Comment on “Absolute and Convective Instabilities in Nonlinear Systems”

When a spatially extended system goes unstable, the ensuing dynamics depends sensitively on whether the system is convectively unstable (in which case perturbations grow in time but are convected away fast enough that they die at each fixed position in the lab frame considered) or absolutely unstable (in which case there exists a perturbation and a location where the perturbation does not decay). The distinction between the two cases for infinitesimal disturbances is well understood; such a linear stability analysis captures most of the essential physics near a supercritical (continuous) bifurcation. Recently, Chomaz [1] studied the nonlinear convective (NLC) versus absolute (NLA) instability near a subcritical (discontinuous) bifurcation for a simple equation that derives from a free-energy-like (Lyapunov) function. The purpose of this Comment is to point out that the case studied by Chomaz is quite restrictive, since it relies on the existence of a unique front separating the basic state from the bifurcating state. In the general case there is a continuum of bifurcating states and an ensuing continuum of fronts, so the problem of selection must be faced. The situation was discussed earlier by two of us [2] in a general investigation of front and pulse propagation near subcritical bifurcations. The extension to systems governed by a Lyapunov function is particularly relevant for the study of nonlinear stability of open hydrodynamic flows or of systems with traveling waves.

As a simple model for dynamics near a subcritical bifurcation, Chomaz [1] studied the real equation

\[ \partial_t A + u_0 \partial_x A - \delta^2 A + \mu A + A^3 - A^5. \]  

The nonlinear stability properties depend on the response to disturbances of finite extent and amplitude. For \(-\frac{1}{4} < \mu < 0\) Eq. (1) admits two homogeneous stable states, \(A_0 = 0\) and \(A_2 \neq 0\). To study the nonlinear stability of the \(A_0\) state it suffices to consider a front solution joining the state \(A_2\) for \(x \to -\infty\) with the state \(A_0\) for \(x \to \infty\), in the symmetrical \((U_0 = 0)\) frame where the \(U_0 \partial_x A\) term is absent. If the front speed \(v\) of this solution is negative, an isolated droplet of the \(A_2\) state in a background of the \(A_0\) state shrinks; hence the \(A_0\) state is stable. If \(v\) is positive, \(A_2\) droplets grow and the \(A_0\) state is (nonlinearly) unstable. Since for \(U_0 = 0\), Eq. (1) is governed by a Lyapunov function \(L = \int dx (\partial_x A)^2 / 2 - \mu A^4 / 4 + A^6 / 6\), the sign of \(v\) depends on the relative magnitude of \(L(A_0)\) and \(L(A_2)\), and \(v = 0\) for \(\mu = \mu_M = -\frac{1}{16}\) where \(L(A_0) = L(A_2)\). In the unstable domain \(\mu > \mu_M\) the instability in the \(U_0\) frame is convective (NLC) for \(v \sim U_0 < 0\), and absolute (NLA) for \(v \sim U_0 \geq 0\).

When a Hopf bifurcation to traveling waves occurs, the amplitude dynamics near a subcritical bifurcation can be modeled by an extension of (1), the complex Ginzburg-Landau equation, which in the symmetrical \((U_0 = 0)\) frame reads

\[ \partial_t A = (1 + i c_1) \partial_x^2 A + \mu A + (1 + i c_3) A |A|^2 + (-1 + i c_5) A |A|^4. \]  

Here \(A\) is the complex valued amplitude, and the \(c_i\)'s are real parameters associated with the linear \((c_1)\) and nonlinear \((c_3, c_5)\) dispersion. Equation (2) cannot be derived from a Lyapunov function, and contrary to (1) has a continuum of bifurcating states.

The surprising finding of Ref. [2] is that the stability properties of the state \(A_0\) are largely determined by the existence or absence of an exact nonlinear front solution with speed \(v^* (\mu, c_1, c_3, c_5)\) that increases for increasing \(\mu\) and is zero for \(\mu = \mu_2 (c_1, c_3, c_5)\). It is found [2] that either (a) this front solution exists and has positive \(v^*\) for some range \(\mu > \mu_3\) with \(\mu_3 < 0\); (b) for all \(\mu < 0\) the front speed is negative (i.e., \(\mu > 0\)); or (c) for \(\mu > 0\) no nonlinear front solution exists.

In case (a) the behavior for \(\mu > \mu_3\) is similar to that found in the real equation when \(\mu > \mu_M\). The state \(A_0\) is unstable, and the instability is NLA for \(v^* - U_0 < 0\) and NLC for \(v^* - U_0 < 0\). For \(\mu \leq \mu_3\), on the other hand, typically stationary pulse solutions exist, over a range \(\mu_2 < \mu < \mu_3\), so although \(v^* < 0\), the state \(A_0\) remains unstable. Since the pulse velocity is in general zero, the instability is NLC for any \(U_0 > 0\). For \(\mu > \mu_2\) the state \(A_0\) is stable. In case (b) the pulse region extends up to \(\mu = 0\), and for \(\mu > 0\) the stability properties are similar to those of a supercritical bifurcation with a front velocity \(v \propto \sqrt{\mu}\). For case (c) less is known, but chaotically spreading front solutions as well as pulses have been found [2]. In some experiments [3], the latter structures help stabilize a system by absorbing small perturbations that are convected into them. It is an open question which regime is relevant for planar Poiseuille flow, where \(c_1 \approx 0.4\) and \(c_3 \approx 0.4\) but \(c_5\) is not known.

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