Nonlocality of high-dimensional two-photon orbital angular momentum states

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We propose an interferometric method to investigate the nonlocality of high-dimensional two-photon orbital angular momentum states generated by spontaneous parametric down conversion. We incorporate two half-integer spiral phase plates and a variable-reflectivity output beam splitter into a Mach-Zehnder interferometer to create entangled photon pairs. This setup enables testing the nonlocality of high-dimensional two-photon states by repeated use of the Clauser-Horne-Shimony-Holt inequality.

I. INTRODUCTION

Entangled qubits play a key role in many applications of quantum information [1] and quantum cryptography [2]. An example of a qubit is the polarization state of a photon. More generally, a qudit is a quantum system whose state lies in a \( d \)-dimensional Hilbert space. The higher dimensionality implies a greater potential for applications in quantum information processing and this explains the continuously growing interest in methods for creating entangled qudits.

Among these methods, spontaneous parametric down conversion (SPDC) appears to be the most reliable one for creating entangled photon pairs [3]. Recently, several techniques have been used to create entangled qudits from down-converted photons. For example, conservation of orbital angular momentum (OAM) in SPDC has been used to create entangled states with \( d=3 \) [4,5], and a time binning method was employed to realize states with \( d=11 \) [6]. Recently, spatial degrees of freedom in SPDC [7] have been exploited to demonstrate entanglement for the cases \( d=4,8 \) [8] and \( d=6 \) [9].

It is well known that useful high-dimensional entanglement can be witnessed by violation of Bell-type inequalities [10], which also furnish a test of nonlocality for a quantum system. However, tests of \( d \)-dimensional inequalities for bipartite quantum systems require the use of at least \( 2d \) detectors, which becomes exceedingly difficult (if not impossible) for large \( d \).

In a previous paper [11] we proposed an experiment to show the entanglement of high-dimensional two-photon OAM states, with two detectors only. This scheme indeed allows us to verify the existence of high-dimensional non-separability, as demonstrated by our subsequent experimental results [12]. In Ref. [11] we went on to use a two-dimensional Bell inequality to check the nonlocality of our OAM-entangled photons. In the meantime we have realized that this implicitly assumes dichotomic variables, a condition that was not fulfilled by the scheme proposed in Ref. [11].

In the present paper, we propose an experimental scheme to explicitly test the nonlocality (namely, the useful entanglement) of very-high-dimensional two-photon OAM states (\( d \approx \infty \)), by using just four detectors. The advantages of our method with respect to those using \( 2d \) detectors are obvious for \( d \geq 2 \). Additionally, we stress that the scheme we propose is designed to realize dichotomic observables. The idea is first to project the infinite-dimensional two-photon state onto several different four-dimensional subspaces (in order to select different four-dimensional two-photon states), and then to apply the Clauser-Horne-Shimony-Holt (CHSH) inequality [13] to each selected state. It is not obvious a priori whether such a scheme will work or not. In fact several legitimate questions can be raised: (i) Does this dimensional reduction spoil the entanglement of the two-photon state? (ii) Do selected four-dimensional states maximally violate the CHSH inequality? (iii) Are distinct four-dimensional subspaces equivalent? In the rest of this paper we will address these questions.

II. THE PROPOSED EXPERIMENT

As shown in Fig. 1, a nonlinear crystal yields OAM-entangled photon pairs, and the two photons (say \( a \) and \( b \)) are fed into two balanced Mach-Zehnder interferometers which are shown in detail in Fig. 2. Each Mach-Zehnder MZ\(_x\) \((x=a,b)\) is made of a 50-50 input beam splitter (BS)
and a variable-reflectivity output beam splitter (VBS). We denote with \( t_z \) and \( r_z \) the transmission and reflection coefficients of each VBS, and assume \( t_z = \cos \theta_z, \quad r_z = \sin \theta_z \), where \( x = a, b \) and \( \theta_z \in [0, 2\pi) \). The role of the VBS in such a scheme is that of a "channel selector" which can change the relative weight of the two arms of the interferometer. Each VBS can be easily realized, for example, by exploiting the polarization degrees of freedom of the SPDC photons. Type I crystals emit photon pairs with a well-defined linear polarization [14]. Then, the combination of a half-wave plate before the Mach-Zehnder and a polarizing beam splitter as output BS of the same interferometer realizes the desired VBS. Another possibility is to use a Fabry-Pérot étalon whose mirror separation can be varied, to realize a so-called "Lorentzian beam splitter" [15], which acts as a VBS.

In channel 1 of interferometer MZ\(_a\) there is a spiral phase plate (SPP) [16] with step index \( L \) oriented at \( \alpha \) (see Fig. 3), while in channel 2 there is a SPP with the same step index but oriented at \( \alpha + \pi \). When the step index is half-integer, that is when \( L = \ell + 1/2, (\ell = 1, 2, \ldots) \), these two antiparallel geometrical orientations (\( \alpha \) and \( \alpha + \pi \)) define, in combination with single-mode fibers (see below), two orthogonal spatial modes [11]. Similarly, in channel 1 of interferometer MZ\(_b\) there is a spiral phase plate (SPP) with negative step index \( -L \) oriented at \( \beta \), while in channel 2 there is a SPP with the same step index but oriented at \( \beta + \pi \).

Each output port of the interferometers is coupled to a single-mode fiber which sustains the Laguerre-Gaussian mode \( L_{p_0}^{l=0} \). When a photon in the arbitrary state \( |\xi\rangle \) is coupled to such a single-mode fiber, the fiber projects the input state of the photon on the Laguerre-Gaussian state \( |l=0, p=0\rangle = |0, 0\rangle \) with probability \( |\langle 0, 0 |\xi\rangle|^2 \). The output port of each fiber is coupled with a single-photon detector. Finally, each photon propagates from the crystal to the single-mode fibers through a suitable system of lenses (not shown in Fig. 1), which images the twin photons from the crystal to the SPPs, and from the SPPs to the input facets of the fibers. In this way, free-space propagation effects reduce to an azimuthal-independent longitudinal phase factor.

### III. THE MACH-ZEHNDER INTERFEROMETER

Each photon enters the Mach-Zehnder interferometer through a single input port, say "port 1." The quantum state of the down-converted photon pair at the entrance of both interferometers, can be written as [17]

\[
|\Psi^{(m)}\rangle \propto \int d^2x \Lambda_{r}(r) \hat{a}_1^\dagger(x) \hat{b}_1^\dagger(x)|0\rangle \tag{1}
\]

where \( \Lambda_{r}(r) = L_{0}^{r}(r,w) \) describes the transverse profile of the pump beam, \( r = |x| \), and \( w \) is the pump beam waist. The entangled photons cross both Mach-Zehnders, and are eventually detected. After a lengthy but straightforward calculation, it is possible to show [18] that the probability \( P_{ij}(\theta_x, \theta_y) \) that the detector \( D_{ai} \) fires in coincidence with the detector \( D_{bj} \) is given by

\[
P_{ij}(\theta_x, \theta_y) \propto |\langle 0, 0 |\hat{U}_i(\alpha, \theta_x) \otimes \hat{U}_j(\beta, \theta_y)|\Psi^{(m)}\rangle|^2, \tag{2}
\]

where

\[
\hat{U}_i(\chi, \theta_x) = \sum_{j=1}^{2} R_{ij}(\theta_x) \hat{S}(\chi), \quad i = 1, 2, \quad \chi = a, b, \tag{3}
\]

is the operator representing the propagation of a photon through the channel \( "i" \) of MZ\(_x\), and \( \hat{S}(\chi) \) is the quantum-mechanical operator representing a half-integer SPP oriented at angle \( \chi_1 \) [18], where \( \chi_1 = \chi + (j-1)\pi \), with \( \chi = \alpha, \beta \). Finally, we introduced

\[
R(\theta_x) = \begin{pmatrix} \cos \theta_x & -\sin \theta_x \\ \sin \theta_x & \cos \theta_x \end{pmatrix}, \quad (x = a, b), \tag{4}
\]

as the orthogonal matrix representing the VBS. Explicit expressions for \( P_{ij}(\theta_x, \theta_y) \) are given in [18]. For our present purpose it is important to note that \( P_{ij}(\theta_x, \theta_y) \) satisfies the no-signaling conditions

\[
\sum_{j=1}^{2} P_{ij}(\theta_x, \theta_y) = P_i(\theta_x), \quad \sum_{i=1}^{2} P_{ij}(\theta_x, \theta_y) = P_j(\theta_y), \tag{5}
\]

for a bipartite, \( 2 \times 2 \) dimensional system. From Eqs. (2) and (3) it follows that when a coincidence detection
orthogonality relations
So, we have two parties
we can see that, e.g., the basis
the quantum nonlocality of the input state
larly, Bob can choose between
and \( S(\alpha_j) = \hat{S}(\alpha_j)|0, 0\rangle \). In a similar manner we define
From the orthogonality relations \[19\]
\[ \langle S(\alpha_j)|S(\alpha_j)\rangle = \delta_{ij} = \langle \hat{S}(\beta_j)|\hat{S}(\beta_j)\rangle, \]
it follows that \( \{|S(\alpha)|,|S(\alpha+\pi)\rangle \} \) and \( \{|\hat{S}(\beta)|,|\hat{S}(\beta+\pi)\rangle \} \)
form an orthogonal two-dimensional basis for the photons \( a \) and \( b \), respectively. Equations (4) and (6) show that the state
\( |u_i(\alpha, \theta_i)|u_j(\beta, \theta_j)\rangle \) onto which the initial state \( |\Psi^m\rangle \) is projected, remains confined to the four-dimensional two-photon subspace spanned by the basis \( \{|S(\alpha_i)|,|\hat{S}(\beta_j)\rangle \} \), \((i, j = 1, 2)\)
when the VBS’s “angles” \( \theta_a \) and \( \theta_b \) are varied. Moreover, we can see that, e.g., the basis \( \{|S(\alpha)|,|S(\alpha+\pi)\rangle \} \) defines a
dichotomic subspace, as the basis \( \{|H\rangle,|V\rangle\} \) does in polarization space. It is clear then that, when we choose a pair \((\alpha, \beta)\)
of SPP’s orientations, we uniquely fix a four-dimensional two-photon subspace.

IV. ADDRESSING THE QUANTUM NONLOCALITY
At this point we know how to calculate the coincidence probabilities \( P_{ij}(\theta_a, \theta_b) \) from the state \( |\Psi^m\rangle \) at the output of both interferometers. However, to proceed further and test the quantum nonlocality of the input state \( |\Psi^m\rangle \), we have to specify our scenario more precisely. We have two parties, say
Alice and Bob, who share the two-photon entangled state \( |\Psi^m\rangle \) given in Eq. (1). Each one of the two entangled photons belongs to an (in principle) infinite-dimensional Hilbert space. Alice and Bob each have a measuring apparatus: \( M_a \) and \( M_b \) respectively. Each apparatus \( M_i \) \((i = a, b)\) consists of a two-channel Mach-Zehnder interferometer MZ \(_\alpha\), with a parameter \( \theta_i \) at the experimenter’s disposal, followed by two (one per channel \( i = 1, 2\)) single-mode fibers \( F_i \). The output ports \( i = 1, 2\) of each \( M_i \) are monitored by two detectors \( D_{1i} \) and \( D_{2i} \) respectively. We stress that in this scenario the SPP rotation angles \( \alpha \) and \( \beta \) are not experimental “knobs” that are changed during an experiment. Different pairs \( \{\alpha, \beta\} \) define different experiments which use the same initial two-photon entangled state \( |\Psi^m\rangle \). In analogy with the polarization case, Alice can choose between two different measurements, say \( A \) and \( A' \), corresponding to two different choices for the varying-beam-splitter “angles” \( \theta_a \) and \( \theta_a' \), respectively. Similarly, Bob can choose between \( B \) and \( B' \), corresponding to \( \theta_b \) and \( \theta_b' \), respectively. Each time Alice and Bob perform a measurement, \( M_i \) \((i = a, b)\) gives the string \( \{x_1, x_2\} \), where \( x_i = 1 \) when the detector \( D_{1i} \) fires and \( x_i = 0 \) when it does not. So, we have two parties (Alice and Bob), two measurements \( \{\theta \_a \text{ and } \theta \_a'\} \) per party, and two possible outcomes \( \{\{1, 0\}\} \) and \( \{\{0, 1\}\} \) per measurement for each party. This situation is usu-
ally indicated as a \( d \times N_a \times N_b = 2 \times 2 \times 2 \) Bell scenario. For this case, as is well known \[20\], the most important test of nonlocality is the CHSH inequality \[21\]
\[ S = |E(\theta_a, \theta_b) - E(\theta_a', \theta_b') + E(\theta_a, \theta_b') - E(\theta_a', \theta_b)| \leq 2, \]
where, in our notation, \( E(\theta_a, \theta_b) \) is given by
\[ P_{11}(\theta_a, \theta_b) - P_{12}(\theta_a, \theta_b) - P_{21}(\theta_a, \theta_b) + P_{22}(\theta_a, \theta_b), \]
\[ P_{11}(\theta_a, \theta_b) + P_{12}(\theta_a, \theta_b) + P_{21}(\theta_a, \theta_b) + P_{22}(\theta_a, \theta_b). \]
We first choose as a special case a common orientation \( \alpha = \beta \) for the SPPs for the two photons. It is then straightforward to show that \( E(\theta_a, \theta_b) = \cos[2(\theta_a - \theta_b)] \) and,
with the particular choice of varying-beam-splitter angles \( \theta_a = 0, \theta_a' = \pi/4 \), \( \theta_b = \pi/8, \theta_b' = 3\pi/8 \), we achieve the maximum violation \( S = 2\sqrt{2} \) of the CHSH inequality. This result is valid for all values of \( \alpha \). For this special case, we find thus the same result as one would achieve describing an experiment involving dichotomic variables, as in the case of polarization-entangled two-photon states. However, unlike the polarization case, here we have an additional parameter at our disposal, namely the SPP orientation angle \( \alpha \).
Next, we pass to the more general case \( \alpha \neq \beta \). For this case we have to use numerical methods. We found, by numerical search, many pairs \( \alpha \neq \beta \) which produce violation close to \( 2\sqrt{2} \). This result is quite interesting since it is a signature that the entanglement of the photon pair may survive this “dimensional reduction” even when different sub-
spaces (viz., different degrees of freedom) are tested. Now, provided that the state vectors \( \{|S(\chi)|,|S(\chi+\pi)\rangle \}
\( |S(\chi')|,|S(\chi'+\pi)\rangle \), \( |S(\chi'')|,|S(\chi''+\pi)\rangle \), \ldots \) \( |\chi = \alpha, \beta\rangle \) are chosen to be linearly independent, we can extend the CHSH test to the \( N \) pairs \( \{(\alpha, \beta), (\alpha', \beta'), (\alpha'', \beta''), \ldots, (\alpha^{(N)}, \beta^{(N)})\} \)
defining \( N \) pairs of two-dimensional subspaces whose union defines a \( 2N \times 2N \) two-photon subspace. In this way we can demonstrate the nonlocal nature of the high-dimensional two-photon OAM-entangled states.
Let us compare our results with the questions (i–iii) posed in the Introduction. From an initial entangled \( \infty \)-dimensional state \( |\Psi^m\rangle \) we obtain entangled four-dimensional states; each dimensionally reduced state is maximally entangled; all four-dimensional subspaces are, in this sense, equivalent. All questions posed in the Introduction have thus been positively answered.

V. CONCLUSIONS
In this paper we proposed an experimental setup to investigate the nonlocality (viz., the degree of useful entangle-
ment) of very high-dimensional two-photon OAM-entangled states, by using four detectors only. We use a pair of modi-
fied Mach-Zehnder interferometers as OAM analyzers. They reduce the effective dimensionality of the two-photon Hilbert space from \( \infty \) to 4. This entanglement-preserving dimensional reduction permits us to check the nonlocality of the two-photon state with a \( 2 \times 2 \times 2 \) inequality \[20\]. In this way we find the maximum violation \( 2\sqrt{2} \) of the CHSH inequality
for any four-dimensional two-photon subspace we choose. Moreover, because of the strict analogy between our four-dimensional two-photon sub-spaces and four-dimensional two-photon polarization space, other interesting experiments (e.g., teleportation of spatial degrees of freedom) can be implemented by using our scheme.

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