1. INTRODUCTION

While the orbital angular momentum (OAM) of light is often viewed as a property of a classical beam,\textsuperscript{1-5} it has recently received much attention at the quantum, i.e., single-photon, level.\textsuperscript{6-14} This is a consequence of the realization that OAM promises to provide a truly multidimensional alphabet for the field of quantum information\textsuperscript{11} in the form of a quNt, i.e., an N-dimensional quantum system. The OAM of a photon in its propagation direction is usually characterized by the integer $\ell$ (in units of $\hbar$), meaning that the expectation value of the photon's OAM equals $\ell \hbar$. Thus an absorber placed in the path of such a beam will acquire an angular momentum of $\ell \hbar$ times the number of absorbed photons. The integer $\ell$ enumerates the OAM states of the photon and thus the quNt's levels.

The most widespread method of generating OAM in an optical beam, and for that matter in a beam of photons, is by imprinting one or more vortices on its transverse field distribution. The vortex charge $Q$ of the resulting beam is defined as

$$Q = \frac{1}{2\pi} \oint \phi \, d\phi,$$

where $\phi$ is the phase of the field, and the closed path goes through points where $\phi$ is analytical and encircles all imprinted vortices; the vortex charge $Q$ is integer valued.

For analytical input beams the output beam can also be described analytically when the vortex-imprinting device has a transmission function $t(r, \theta) = r \exp(i\theta) - r_0 \exp(i\theta_0)$ results in a superposition of two LG modes, $u_{00}(r, \theta)$ and $u_{10}(r, \theta)$.\textsuperscript{16} In practice, however, generation of helical beams is accomplished by passing a fundamental Gaussian beam through a device that modifies only the phase and not the amplitude, i.e., the transmission is function $t(r, \theta) = \exp[i\phi(r, \theta)]$. A vortex can be obtained by choosing, for instance, $\phi(r, \theta) = \theta$ so that $t(r, \theta) = \exp[i\theta]$. We will refer to this type as a phase vortex, also known in the literature as a point vortex.\textsuperscript{20-22} It shares the phase structure but not the amplitude structure with the analytical vortex. An incident fundamental Gaussian beam results in superposition of many LG beams, all with the same OAM.\textsuperscript{4} A popular device for generating pure phase vortices is the hologram with a dislocation in its center,\textsuperscript{23,24} also called the fork hologram.

The topological charge of the imprinted vortex depends on the strength of the dislocation and the diffraction order of the hologram. When the dislocation is well centered with respect to the cylindrically symmetric incident beam, the diffracted beams are also cylindrically symmetric and carry no net transverse momentum relative to their symmetry axis. When the OAM is calculated around the symmetry axis of each beam, its value per photon in units of $\hbar$ represents the intrinsic OAM,\textsuperscript{25} and is simply equal to the vortex charge $Q$ multiplied by the order number $m$.

When the hologram, and thus the dislocation, is not centered with respect to the beam, thus generating an off-axis vortex, the far-field profile will, in general, not have cylindrical symmetry.\textsuperscript{11,22} The propagation axis of the diffracted beam, which is defined as the axis with respect to which the field has no net transverse momentum, will change direction when the dislocation is displaced. It is, however, precisely this propagation axis that is required to calculate the intrinsic OAM of a beam.\textsuperscript{25} Therefore a calculation of the intrinsic OAM as a function of the hologram displacement becomes cumbersome because for each displacement, one first has to find the propagation axis of the beam.

Intrinsic orbital angular momentum of paraxial beams with off-axis imprinted vortices

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We investigate the orbital angular momentum (OAM) of paraxial beams containing off-axis phase dislocations and put forward a simple method to calculate the intrinsic orbital angular momentum of an arbitrary paraxial beam. Using this approach we find that the intrinsic OAM of a fundamental Gaussian beam with a vortex imprinted off axis has a Gaussian dependence on the vortex displacement, implying that the expectation value of the intrinsic OAM of a photon can take on a continuous range of values (i.e., integer and noninteger values in units of $\hbar$). Finally, we investigate, both numerically and experimentally, the far-field profiles of beams carrying half-integer OAM per photon, these beams having been created by the method of imprinting off-axis vortices. © 2004 Optical Society of America

In this paper we will present an alternative method for calculating the OAM of a beam that is diffracted from an off-axis-positioned fork hologram. Having introduced that method, we apply it to the case in which the input beam is a fundamental Gaussian and show that the intrinsic OAM of a diffracted beam has a Gaussian dependence on the transverse displacement of the dislocation in the hologram relative to the center of the input beam. The expectation value of the intrinsic OAM per photon in the $m$th diffraction order of a hologram with integer dislocation strength $Q$ can thus take on any value $|\ell| \hbar < |m|Q|\hbar$ (while its eigenvalue spectrum remains integer). Throughout this paper we will refer only to the expectation value of the OAM; so, in spite of our using the word “photon,” the treatment is essentially classical, and “photon” is only meant as a convenient energy unit.

Note that in the present scheme, the noninteger character of the OAM is not imprinted in the fork hologram but is caused by its off-axis use. Beams carrying noninteger OAM can also be generated by inserting a phase-modifying device on axis in a Gaussian beam; in that case the device must carry a mixed screw-edge dislocation and therefore a fixed, noninteger dislocation strength. Examples of these are spiral phase plates and fork holograms with half-integer dislocation strength.4,26,27 The method outlined in the present paper has thus added flexibility by the ability to simply tune the OAM. To do so, we introduce a more general criterion for determining the intrinsic nature of OAM. Finally, we illustrate our theoretical work with experimental and calculated far-field patterns of beams that have OAM multiply imprinted in units of $1/2\hbar$ per photon by stacking displaced holograms in the same Fourier plane.

2. OPTICAL VORTICES AND ORBITAL ANGULAR MOMENTUM

Paraxial optical beams carry both spin and orbital angular momentum, where the former is connected to the polarization of the light and the latter to the transverse field distribution.1,2 For a classical, paraxial, monochromatic laser momentum, where the former is connected to the polarization of the light and the latter to the transverse field distribution.1,2 The expectation value of the intrinsic OAM per photon in the $m$th diffraction order of a hologram with integer dislocation strength $Q$ can thus take on any value $|\ell| \hbar < |m|Q|\hbar$ (while its eigenvalue spectrum remains integer). Throughout this paper we will refer only to the expectation value of the OAM; so, in spite of our using the word “photon,” the treatment is essentially classical, and “photon” is only meant as a convenient energy unit.

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2. OPTICAL VORTICES AND ORBITAL ANGULAR MOMENTUM

Paraxial optical beams carry both spin and orbital angular momentum, where the former is connected to the polarization of the light and the latter to the transverse field distribution.1,2 For a classical, paraxial, monochromatic (frequency $\omega$) beam with uniform polarization, indicated by the transverse vector $\mathbf{a}$, the electric field can be written as

$$ E(r, t) = a u(r) \exp(ikz - i\omega t) + c.c. \quad (2) $$

As OAM is a constant of propagation, we can, for the purpose of calculating the OAM, consider $u(r)$ in any transverse plane. We choose $z = 0$ so that we can write $u(r, \theta, 0) = u(p)$, where $p = (r, \theta)$ is the transverse position vector. The density of the transverse linear momentum of a linearly polarized beam is given by

$$ p(p) = \frac{\varepsilon_0}{i\omega} \left[ u^*(p) \nabla u(p) - c.c. \right], \quad (3) $$

where $\nabla$ is the transverse gradient operator. The $z$ component of the OAM per unit length can be calculated as

$$ L = \int [p \times \mathbf{p}(p)] dp = \frac{2\varepsilon_0}{\omega} \int \left[ u^*(p) \left( \frac{\partial}{\partial \theta} \right) u(p) \right] dp, \quad (4) $$

where in polar coordinates $p \times \nabla = \partial / \partial \theta$. As Eq. (3) reveals, the momentum density and thus the OAM are determined by the phase gradient of the field distribution $u(p)$. The OAM per photon is then

$$ \ell \hbar = \frac{\hbar \omega}{2\varepsilon_0} L \quad (5a) $$

$$ = \int u^*(p) \frac{\hbar}{i} \left( \frac{\partial}{\partial \theta} u(p) \right) dp \int |u(p)|^2 dp, \quad (5b) $$

where the denominator in Eq. (5a) is the field energy per unit length. Note that Eq. (5b) has the appearance of the quantum-mechanical expression for the expectation value of the $z$ component of the OAM of a wave function.

A convenient basis for describing the OAM of a paraxial beam is the complete orthonormal set of LG modes.1,28 These cylindrically symmetric modes possess, relative to their symmetry axis, an OAM per photon that is integer valued in units of $\hbar$. In the plane $z = 0$ an LG mode can be expressed as

$$ u_{\ell p}^{LG}(r, \theta) = C_{\ell p}^{LG}(r \sqrt{2/w_0}) L_p^\ell(2r^2/w_0^2) \times \exp(-r^2/w_0^2) \exp(i\ell\theta), \quad (6) $$

where $C_{\ell p}^{LG}$ is a normalization constant, $L_p^\ell(x)$ is a generalized Laguerre polynomial,29 the integers $l \in \{-\infty, \ldots, \infty\}$ and $p \in \{0, \ldots, \infty\}$ are indices characterizing the transverse profile of the mode, and $w_0$ is the beam waist radius. The vortex contained within an LG mode lies at the center; it is caused by the exponential phase factor and has a charge equal to $l$ [see Eq. (6)]. The OAM per photon in units of $\hbar$ contained in this field is also given by $l$ (Refs. 1 and 2) and is therefore always integer. Information regarding the radial profile of the mode is contained in the index $p$.

To calculate the OAM of a beam with any number of on- or off-axis phase vortices, one usually makes a mode decomposition,

$$ u(p) = \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} C_{\ell p} u_{\ell p}^{LG}(p), \quad (7) $$

where $C_{\ell p}$ are the coefficients of the LG components as

$$ C_{\ell p} = \int_0^{2\pi} \int_0^\infty u(p) \bar{u}_{\ell p}^{LG}(p) \hbar dp. \quad (8) $$

The OAM per photon in units of $\hbar$ carried by the field $u(p)$ is given by

$$ \ell = \sum_{l=-\infty}^{\infty} \sum_{p=0}^{\infty} l |C_{\ell p}|^2, \quad (9) $$

where the $z$ axis coincides with the symmetry axis of the LG modes. It is thus immediately clear that for a beam
carrying an off-axis vortex, the expectation value of the OAM per photon is, in general, noninteger with respect to the chosen $z$ axis. This also shows that vortex charge and OAM are not the same concept. The topological charge of a vortex is completely independent of the chosen reference system and is thus independent of its position, whereas this is not always true for OAM. To further stress the difference, it is important to realize that an optical beam can in fact carry OAM in the absence of any topological charge.\cite{30}

\section{The Fork Phase Hologram}

A fork phase hologram is basically a phase grating, carrying a fundamental reciprocal lattice vector $\mathbf{g}$, with a phase distribution $\exp(iQ\theta)$ superimposed, $Q$ being the topological charge to be imprinted in the first-order diffracted field. The factor $\exp(iQ\theta)$ manifests itself as a fork-like feature in the phase pattern of the hologram, giving rise to a screw dislocation in the nonzero diffraction orders. Figure 1 shows a binary variant of such a fork phase hologram where the black and white in the figure indicate the two values of the phase shift, usually differing by $\pi$. When a fundamental Gaussian beam is perpendicularly incident on such a hologram, the various diffraction orders will pick up the azimuthal component of the hologram’s phase distribution such that the $m$th diffraction order will carry a phase factor $\exp(imQ\theta)$. When the incident beam and hologram are centered with respect to each other, each diffraction order is cylindrically symmetric around its own propagation axis. In this case the propagation axis, i.e., the axis with respect to which the beam has zero transverse momentum, coincides with the diffraction axis $z_m$. The direction of the latter axis is determined solely by the reciprocal lattice vector $\mathbf{g}$ of the hologram and the wavelength of the incident light, and is influenced neither by the hologram’s topological component nor by its position in the incident beam.

For a calculation of the OAM of a diffracted beam, we will use a coordinate system in which the $z$ axis is aligned with the diffraction axis introduced above. In this coordinate system the input field becomes

$$u^\text{in}(\mathbf{r}) \rightarrow u^\text{in}_m(\mathbf{r})\exp(-im\mathbf{g} \cdot \mathbf{r}),$$

where the vector $\mathbf{g}$ is the aforementioned reciprocal lattice vector of the grating structure in the binary hologram.

The phase fronts of the diffracted beams are helical in this coordinate system as a consequence of the dislocation in the hologram:

$$u^\text{out}_m(\rho) = C_mu^\text{in}_m(\rho)\exp(imQ\theta),$$

where the real constant $|C_m|^2$ represents the diffraction efficiency into the $m$th order, and we have assumed that the dislocation is positioned in the center of the beam at $\rho = 0$. Applying Eq. (5b) to Eq. (11) yields an OAM per photon equal to $mQ\hbar$.

\section{Off-Axis Vortices}

When the fork hologram is displaced over a distance $|\rho_v|$ in the transverse plane of the incident Gaussian beam, the centers of intensity of the diffraction orders in the far field shift. The direction of the propagation axis, i.e., the zero-transverse-momentum axis, is thus changed, since it is defined as the axis that goes through the center of intensity of the field in all transverse planes. As a result, the transverse momentum with respect to the previously defined diffraction axis is nonzero,\textsuperscript{7,25} as indicated in Fig. 2. The fact that the orientation of the propagation axis depends on the displacement $\rho_v$ of the fork hologram implies that to calculate the intrinsic OAM of the output beam for different values of $|\rho_v|$, one would first have to determine the propagation axis for each value of the displacement.

Here we will show that to calculate the intrinsic OAM of the diffracted beam, one may always use the diffraction axis, conditional on a simple constraint being satisfied. The advantage of this approach is that the diffraction axis...
is fixed in a given experiment and is thus independent of the displacement of the hologram.

In classical mechanics, the intrinsic angular momentum of a rigid body is obtained by taking a frame of reference in which the origin lies at the center of mass. Obviously the $z$ component of the angular momentum is then also intrinsic. Tilting the $z$ axis at an angle $\alpha$ changes the calculated value of that component by a factor $\cos \alpha$. Thus when $\alpha$ is small, the $z$ component of the angular momentum is essentially unaffected.

This approach can also be applied to the $z$ component of the OAM of a beam of light \(^31\) by choosing the origin of the frame at the center of the beam in the near field, while the $z$ axis is chosen to be parallel to the propagation axis of the beam. It has been shown that the latter condition gives the freedom to arbitrarily choose the position of the origin without influencing the value of the $z$ component of the OAM. \(^7,25\) We will show that instead, we may choose to keep the origin fixed in the center of the near-field intensity distribution and tilt the $z$ axis at an angle $\alpha$ to align it with the diffraction axis.

**A. Intrinsic Orbital Angular Momentum of a Paraxial Optical Beam**

Let us assume an arbitrary, monochromatic, paraxial beam of which the amplitude distribution in the plane $z = 0$ is described by the complex function $u(\rho)$, where the coordinate system $S$ has been chosen so that the beam is centered at the origin and does not carry transverse momentum, and

$$\int \rho |u(\rho)|^2 d\rho = 0, \quad \quad (12)$$

$$\int p(\rho) d\rho = 0. \quad \quad (13)$$

Here $\rho$ indicates the transverse position and $p$ is the transverse linear momentum density per unit length as given in Eq. (3). Thus Eq. (12) fixes the origin of the reference frame in the center of the intensity distribution, while Eq. (13) aligns the $z$ axis with the propagation axis of the field.

The $z$ component of the OAM per unit length in coordinate system $S$ is then given by

$$L^0 = \int [\rho \times p(\rho)]_z d\rho. \quad \quad (14)$$

Because of the definition of our coordinate system given in Eqs. (12) and (13) this is the intrinsic OAM. We will now study the effects of tilting and displacing the beam with respect to the reference frame.

When we tilt the field $u(\rho)$ at an angle $\alpha$ with respect to the $z$ axis, we obtain

$$u'(\rho) = u(\rho) \exp(i \mathbf{k}_0 \cdot \rho), \quad \quad (15)$$

neglecting the tilt-induced, second-order deformation of the amplitude profile. Here $\mathbf{k}_0$ represents the tilt, and, in the paraxial approximation, the length of $\mathbf{k}_0$ is equal to $\alpha$ times the beam’s wave vector. Here $\mathbf{k}_0$ is a displacement in $\mathbf{k}$ space due to the tilt. We neglect the tilt-induced second-order deformation of the amplitude profile. The transverse component of the momentum density per unit length of the tilted beam is given by

$$p'(\rho) = \frac{\epsilon_0}{i \omega}[u^*(\rho) \nabla u(\rho) - u(\rho) \nabla u^*(\rho) + 2i \mathbf{k}_0 |u(\rho)|^2]$$

$$= p(\rho) + \frac{2 \epsilon_0}{\omega} \mathbf{k}_0 |u(\rho)|^2. \quad \quad (16)$$

The transverse displacement $\rho_0$ of the field is accounted for by calculating the $z$ component of the OAM with respect to the point $\rho_0$,

$$L' = \int [(\rho - \rho_0) \times p'(\rho)]_z d\rho$$

$$= L^0 - \frac{2 \epsilon_0}{\omega} (\rho_0 \times \mathbf{k}_0)_z \int |u(\rho)|^2 d\rho. \quad \quad (17)$$

where two cross terms have vanished as a result of our initial constraints [see Eqs. (12) and (13)]. We find that generally the OAM $L'$ of the tilted beam with respect to a point at a position $\rho_0$ differs from $L^0$. It is also clear that by a proper choice of $\rho_0$ or $\mathbf{k}_0$, $L'$ can be made equal to $L^0$.

One way to have $L' = L^0$ is to choose $\rho_0$ equal to zero: The OAM should always be calculated with respect to the center of the beam in the near field. In that case the (paraxial) tilt of the beam with respect to the $z$ axis is irrelevant, i.e., the choice of $z$ axis is free.

The second way to have $L'$ equal to the intrinsic momentum is to align the $z$ axis so that it is parallel to the propagation axis of the beam, $\mathbf{k}_0 = 0$. Consequently, the field will not have a net transverse momentum. It then does not matter with respect to which transverse position $\rho_0$ the OAM is considered, as it is found to be intrinsic. This constraint for which the transverse momentum of the beam is required to vanish was put forward earlier by Berry.\(^{25}\)

From the calculation, a third and final method is found, namely, choosing $\rho_0 \parallel \mathbf{k}_0$, independent of the magnitude of each of these vectors.

The first criterion that we introduced ($|\rho_0| = 0$) gives us the freedom to use the diffraction axis for calculating the intrinsic OAM. Because the orientation of the diffraction axis is independent of the displacement of the hologram, we thereby have a very practical method to calculate the intrinsic OAM; this is the one we will use.

**B. Orbital Angular Momentum of a Gaussian Beam with an Off-Axis Phase Vortex**

Having found a simple method to obtain the intrinsic value of OAM, we now proceed to calculate this quantity. We consider the case in which a fundamental Gaussian beam intersects a fork phase hologram in its waist. The incident field is described by the function

$$u^{in}(\rho) = B \exp(-|\rho|^2/w_0^2), \quad \quad (18)$$

where $w_0$ is the waist and where the constant $B$ is the peak amplitude. We thus have chosen a frame in which the position of the origin is fixed at the center of the intensity distribution in the near field. The hologram has its dislocation at a transverse position $\rho_0$ relative to the
beam’s center. We wish to calculate the intrinsic OAM of the $m$th diffraction order, which is the OAM with respect to the center of the beam. We use as integration variable $\rho$ the position with respect to the vortex, so that $\rho + \rho_v$ is the position with respect to the beam. The $m$th-order diffracted beam is given by

$$u_m^{\text{out}}(\rho) = C_m u_m^{\text{in}}(\rho + \rho_v) \exp(i m Q \theta),$$

and the intrinsic OAM is given by

$$L_m^0 = \frac{2 \varepsilon_0}{\omega} \int u_m^{\text{out}}(\rho)[(\rho + \rho_v) \times \nabla]_{z} u_m^{\text{out}}(\rho) d\rho. \quad (20)$$

When Eq. (19) is substituted into Eq. (20), the gradient operator acting on the product $u_m^{\text{in}} \exp(i m Q \theta)$ gives rise to two terms. The contribution to $L_m^0$ from the first term is simply equal to $C_m^2 L_m^0$, which vanishes since the input beam has zero OAM. To calculate the contribution from the gradient of the phase term $\exp(i m Q \theta)$, it is convenient to apply the gradient operator in polar coordinates, which gives the identity

$$[(\rho + \rho_v) \times \nabla]_{z} \exp(i m Q \theta) = i m Q \frac{\rho \cdot (\rho + \rho_v)}{r^2} \times \exp(i m Q \theta), \quad (21)$$

where $\rho = (r, \theta)$ so that $r = |\rho|$ and $r_v = |\rho_v|$. Thus for the intrinsic OAM, we find the integral expression in polar coordinates

$$L_m^0 = \frac{2 \varepsilon_0}{\omega} C_m^2 m Q B^2 \int \int \exp(-2 r^2 + 2 r r_v \cos \theta + r_v^2) w_0^2 \exp(-2 r_v^2 / w_0^2) dr d\theta$$

$$= C_m^2 m Q \frac{\varepsilon_0}{\omega} w_0^2 \pi B^2 \exp(-2 r_v^2 / w_0^2). \quad (22)$$

We thus find that the intrinsic OAM of the output beam has a Gaussian dependence on the vortex displacement. This result is conceptually attractive: For large displacements $r_v$ of the dislocation away from the beam’s center, the intrinsic OAM tends toward the OAM of the incident beam, which is equal to zero in the case considered here. In that limit, the imprinted vortex simply gives rise to a transverse linear momentum in the output beam.

When a beam of photons is diffracted by a hologram carrying an off-axis vortex, applying Eq. (5b) shows that each diffracted photon carries an intrinsic OAM equal to $m Q h \exp(-2 r_v^2 / w_0^2)$. Consequently, the expectation value of the OAM of these photons is no longer necessarily an integer multiple of $h$. For instance, imprinting a vortex with charge unity ($Q = 1$) displaced by an amount $w_0(\ln(2)/2)^{1/2}$ relative to the Gaussian input beam yields an intrinsic OAM per photon equal to $h/2$, i.e., $\ell = 1/2$ in the first diffraction order ($m = 1$); this can easily be found by using Eq. (5b) and Eq. (22) with $L_m^0 = 0$. In the experiments that are discussed below the photons carry an OAM with that expectation value.

5. Beams with $\ell = 1/2$

A. One Off-Axis Vortex

The setup shown in Fig. 3 allows us to generate an $\ell = 1/2$ beam by letting a fundamental Gaussian impinge on a hologram containing a single screw dislocation displaced over a distance $w_0(\ln(2)/2)^{1/2}$ relative to the center of the input beam. Here $w_0$ is the waist of the incident beam. The lens $L_1$ with focal length $f$ assists in imaging the far field of the $\ell = 1/2$ beam, which is the first diffraction order. A calculated near-field intensity profile is seen in Fig. 4(a), where the $\oplus$ symbol indicates the position of the dislocation causing a vortex of charge +1 in the first diffraction order. The far-field intensity profile of this beam at a wavelength of 633 nm can be seen in Fig. 4(b), while the far-field pattern as calculated by taking the Fourier transform of the near field is shown in Fig. 4(c). The $\oplus$ symbol in Fig. 4(c) indicates the location of a +1 charged vortex in the far field. Both the calculated and CCD image are peak normalized to enhance their contrast. The experimental image agrees very well with the simple diffraction calculation. The center of the intensity distribution in Fig. 4(c) does not coincide with the far-field position of the diffraction axis, the latter indicated by the crossing dashed lines; this $\ell = 1/2$ beam is a clear example of a beam for which propagation and diffraction axes are not coincident.

As observed in both numerical calculations and experimental results, we find that the position where the phase

![Fig. 3](image)

![Fig. 4](image)
vortex \( \exp(i m Q \theta) \) was imprinted in the near field does not coincide with that in the far field; it appears as if the position of the vortex rotates over \( \pm 90^\circ \) when the beam propagates from near field to far field. This is similar to the behavior of a fundamental Gaussian with off-axis analytical vortices, i.e., vortices with both amplitude and phase information.

Since we have chosen our frame of reference so that in the near field its origin coincides with the center of the intensity distribution, and the beam is paraxial, we satisfy one of the three constraints as outlined in Subsection 4 A. Therefore the half-integer OAM in this experiment is intrinsic.

B. Two Off-Axis Vortices

Because of applications in quantum information, there is currently considerable interest in beams containing multiple off-axis vortices. Here we will restrict ourselves to the case of two off-axis vortices, imprinted by phase holograms as described in the previous sections, that are displaced in the same or opposite direction relative to a fundamental Gaussian input beam. The displacement of each hologram in units of the beam's waist \( w_0 \) equals \((\ln(2)/2)^{1/2}\). Consequently, each hologram imparts OAM to its first diffraction orders, \( |\ell| = 1/2 \).

In the setup of Fig. 5 the +1st diffraction order of the first hologram acquires \( \ell = 1/2 \). It propagates toward the second hologram, passing through a telescope of unit magnification (lenses \( L_1 \) and \( L_2 \)) and acquires additional OAM by being diffracted at the second hologram. The +1st diffraction order of this second hologram thus has \( \ell = +1/2 + 1/2 = +1 \), while the −1st diffraction order carries \( \ell = +1/2 - 1/2 = 0 \).

The inverting telescope has two effects: (i) It images the near field of the first hologram on top of the second, whereby the two phase dislocations are imprinted in the same transverse plane, and (ii) it inverts the image of the first hologram. In the setup shown, both hologram dislocations are shifted upward relative to the incident beam, thereby suggesting that the phase dislocations become superposed. Because of the image inversion, the imprinted phase dislocations are in fact not superposed but appear on opposite sides of the input beam.

Figure 6 shows the first diffraction orders of the second hologram. Each row shows, from left to right, (i) a calculated near-field distribution, where the imprinted vortices are indicated with the \( \oplus \) and \( \oslash \) symbols and the dashed lines indicate the center of the beam; (ii) an experimentally obtained far-field intensity profile (at \( \lambda = 633 \text{ nm} \); and (iii) a far-field image as calculated by taking the Fourier transform of the theoretical near field. The top row shows the results for the −1st diffraction order of the second hologram. In the far field the transverse intensity distribution forms a peculiar pattern [Fig. 6(b)] that is in excellent agreement with the result of our calculations [Fig. 6(c)]. A calculation of the far-field phase distribution (not shown here) reveals that the two lobes in Fig. 6(c) have opposite phase. The bottom row shows the result for the +1st diffraction order of the second hologram. Here the far field has a total vortex charge equal to 2, an OAM expectation value equal to +1\( \hbar \) per photon, and a dumbbell-like transverse profile [Fig. 6(e)] that matches quite well the result of our calculations [Fig. 6(f)].

When the second hologram in Fig. 5 is shifted downward with respect to the beam, the phase dislocations in the near field of the second hologram will coincide. The results are shown in Fig. 7. The top row shows, for the \( m = −1 \) diffraction order, the calculated near field [Fig. 7(a)], an experimentally obtained far field [Fig. 7(b)], and a far-field profile calculated by Fourier transforming the calculated near field [Fig. 7(c)]. The superposition of the two oppositely charged vortices is indicated by the \( \oslash \) sym-
The two vortices are combined to form one vortex of charge 51.

Fig. 7. Two vortices are imprinted on top of each other in a fundamental Gaussian beam. In the top row the vortices have opposite charge, thus effectively annihilating each other, whereas in the bottom row their charges are positive, thus being summed. In each row the first column shows a theoretical near-field profile where the intersection of the dashed lines indicates the center of the beam and the \( \bigcirc \) (\( Q = 0 \), overlapping oppositely charged vortices) and \( \bigotimes \) (\( Q = 2 \), overlapping equally charged vortices) symbols indicate the approximate positions of the annihilated and summed vortices, respectively. The second column shows the experimentally obtained far-field intensity profiles corresponding to the near fields shown in the first column. The last column shows far-field profiles that have been calculated by Fourier transforming the near fields.

As the two vortices annihilate each other, a fundamental Gaussian is expected, and found, in the far field. The bottom row shows the same images for the \( m = +1 \)st diffraction order. In the theoretical near field the two vortices are combined to form one vortex of charge 2 indicated by the \( \bigotimes \) symbol [Fig. 7(d)]. The field will carry an OAM expectation value equal to \( (1/2 + 1/2) \hbar \) per photon, and the far-field profile [Fig. 7(e)] also carries a single vortex of charge 2, closely resembling the result of our calculations [Fig. 7(f)].

Finally, these experiments show that when two displaced vortices are embedded in the near field of an optical beam, simple arithmetic rules are obeyed with respect to the OAM generated by each individual vortex, the latter being predicted by Eq. (22). Evidently this will hold also for the case where the number of vortices is larger than two.

6. CONCLUSIONS

We have presented a study of the generation of optical OAM by imprinting a pure phase vortex off axis in a fundamental Gaussian beam. In this context we have determined a practical method of finding the intrinsic OAM that is useful for any arbitrary paraxial beam, as one has only to determine the near-field center of the beam’s intensity profile. We find that the intrinsic OAM of such a beam containing a single off-axis vortex is noninteger and decreases exponentially with the square of the displacement of the vortex from the center of the beam in the near field. Note that this approach is fundamentally different from the approach using mixed screw-edge dislocation devices, as can be shown by performing a mode decomposition; with the method outlined here 5 LG components sufficiently (\( \sim 88\% \)) describe the \( \ell = 1/2 \) beam, in contrast to the 11 LG components required when a device with a mixed screw-edge dislocation is used.

We have experimentally implemented our recipe for creating optical beams with noninteger, intrinsic OAM modes, and we have compared the experimental far-field intensity profiles with those calculated by using simple diffraction theory. Additionally, we have experimentally embedded two vortices inside a fundamental Gaussian beam with each generating half-integer OAM, resulting in an optical beam carrying integer OAM. Again, these beams were investigated by comparing far-field diffraction profiles with the results of diffraction calculations, and we found excellent agreement. We have also observed that the propagation of phase vortices seems similar to that of analytical vortices. Finally, we find that the contributions to the intrinsic orbital angular momentum of the individual vortices can indeed be summed to find the total intrinsic OAM of the beam.

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REFERENCES AND NOTES


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31. It is important to realize that separation of the total angular momentum of a beam into a spin part and an orbital part can be done only for the component of the total angular momentum that lies parallel to (or at least makes a paraxial angle with) the propagation axis of the beam.