Wave description of geometric modes of a resonator

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By using both an operator and a geometric argument, we obtain a wave description of geometric modes of a degenerate optical resonator. This is done by considering the propagation of a displaced Gaussian beam inside the resonator. The round-trip Gouy phase, which is independent of the wavelength of the light, determines the properties of the Gaussian eigenmode. The extra freedom in the case of degeneracy allows for the existence of geometric modes. © 2005 Optical Society of America

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1. INTRODUCTION

The propagation of light waves is described by the wave equation, which can be derived from the Maxwell equations. If the propagation direction of a light wave is well determined, which is the case for light beams, the paraxial approximation can be applied, giving rise to the paraxial wave equation, which is similar in form to the Schrödinger equation for a free particle in two dimensions. This suggests that an operator method can be used to describe the propagation of a light beam and obtain the modes of free space. Also, any Gaussian beam transforms into itself after a round trip, determine the round-trip Gouy phase of the Gaussian eigenmode of the resonator. When the Gouy phase is $2\pi K/N$, where $K$ and $N$ are integers, the resonator is $N$-fold degenerate, and any ray retraces itself after $N$ round trips. This defines a geometric mode. Also, any Gaussian beam transforms into itself after $N$ round trips, which follows from the $ABCD$ law. This indicates that a displaced Gaussian beam transforms into itself after $N$ round trips, where the center of the Gaussian beam follows the trajectory of a ray. By using both an operator and a geometric argument, we obtain a wave description of geometric modes by considering the propagation of a displaced Gaussian beam inside an $N$-fold degenerate resonator. The advantage of our description is that there is a clear physical picture, which is lacking in the description of geometric modes by Chen et al., who use an analogy with spin-coherent states.

In Section 2 we develop the operator method for the description of light beams propagating inside an optical system. We introduce the displacement operator, which shifts and tilts a beam, and show that the center of the displaced beam follows the trajectory of a ray. In Section 3 we define ladder operators that generate the fundamental and higher-order Gaussian eigenmodes of the optical system. The evolution of these ladder operators is also governed by the $ABCD$ matrix of the optical system, from which the $ABCD$ law follows immediately. In Section 4 we apply the operator method to obtain the eigenmodes of a two-mirror resonator and consider the case of degeneracy. In Section 5 we obtain a wave description of geometric modes by using a simple geometric argument. In Section 6 we discuss briefly some peculiar resonator configurations, such as the symmetric confocal resonator.

2. OPERATOR DESCRIPTION FOR OPTICAL SYSTEMS

The real electric field of a monochromatic light beam that propagates in the positive $z$ direction is taken as $\text{Re}[\tilde{E}(R,z,t)]$, where $R=(x,y)$ is the transverse coordinate vector. In this expression $\tilde{e}$ is the normalized polarization vector, and $\tilde{E}$ the complex electric field, which is related to the normalized beam profile $u(R,z)$ by

$$ E(R,z,t) = E_0 u(R,z) \exp(ikz - i\omega t), $$

(1)

where $E_0$ is the complex amplitude of the field. The polarization will be assumed to be uniform throughout this paper, and polarization effects are not considered. Equation (1) is inserted into the scalar wave equation, and it is assumed that $|\partial u/\partial z| \ll ku$, which means that the profile $u(R,z)$ varies only slowly with $z$. Then the profile $u(R,z)$ satisfies the paraxial wave equation:

$$ \left( \frac{\partial^2}{\partial R^2} + 2ik \frac{\partial}{\partial z} \right) u(R,z) = 0, $$

(2)

where $\partial^2/\partial R^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The paraxial wave equation is identical in form to the Schrödinger equation for a free particle in two dimensions, where time plays the role of the longitudinal coordinate $z$. This similarity is the starting point for an operator description of the propagation of a monochromatic light beam through an optical system containing Gaussian optical elements that are lossless.
and nonastigmatic. Examples of optical elements are the propagation through vacuum, lenses, and mirrors.

The transverse profile \( u(R, z) = \langle R | u(z) \rangle \) of the light beam corresponds to the time-dependent wave function in quantum mechanics, and \( | u(z) \rangle \) is the “state” of the light beam in the transverse plane \( z \). The propagation through vacuum is described by

\[
| u(z) \rangle = \hat{U}(z) | u(0) \rangle, \quad \hat{U}(z) = \exp \left( -i z \hat{P} \right),
\]

where \( \hat{P} = \hat{p}_x^2 + \hat{p}_y^2 \). The transverse momentum operator \( \hat{P} \) takes the form \(-i \partial / \partial R\) in the coordinate representation.

When passing through a parabolic lens, the transverse profile of the light beam acquires a phase shift that depends quadratically on the transverse coordinate. It is assumed that the lens is thin, so that the transverse beam profile is constant inside the lens. Then the effect of the lens is described by multiplying the transverse beam profile by a parabolic phase factor. We consider only cases where the optical axis of the lens coincides with the \( z \) axis. When the lens is located in the transverse plane \( z \), the state of the light beam \( | u(z) \rangle \) after the lens is expressed in terms of the state \( | u(z) \rangle \) before the lens by

\[
| u(z) \rangle = \hat{U}(f) | u(z) \rangle, \quad \hat{U}(f) = \exp \left( -i k \hat{R}^2 / 2f \right),
\]

where \( f \) is the focal distance of the lens. We mention that Eq. (4) also holds for spherical lenses as long as the approximation of the spherical shape by a parabola is accurate enough within the spot size of the light beam on the lens (which is usually the case for thin lenses). Under this condition the spherical lens can be considered a Gaussian optical element. Now the change of the state of the light beam when going from the input plane to the output plane of the optical system is described by the unitary operator \( \hat{U} \), which is a product of multiples of \( \hat{U}_f \) and \( \hat{U}_l \) in the proper order, according to the arrangement of the lenses in the optical system. For instance, the optical system in Fig. 1, which begins at \( z = 0 \) and ends at \( z = 2L \), has a lens with focal distance \( f_1 \) at \( z = 0 \) and a lens with focal distance \( f_2 \) at \( z = L \). The evolution operator \( \hat{U} \) for this optical system is equal to

\[
\hat{U} = \hat{U}_f(L) \hat{U}_l(L) \hat{U}_f(f_2) \hat{U}_l(L) \hat{U}_f(f_1).
\]

We define the displacement operator by

\[
\hat{D}(a, q) = \exp(ia \hat{x} \hat{R} - iq \hat{x} \hat{R}),
\]

where \( a \) is a position vector in the transverse plane, \( q \) is a transverse momentum vector, and \( a \hat{x} \hat{R} \) indicates the inner product of \( a \) with the transverse coordinate operator \( \hat{R} = (\hat{x}, \hat{y}) \). The significance of the displacement operator becomes clear from its properties:

\[
\hat{D}(a, q) \hat{R} \hat{D}(a, q) = \hat{R} + a,
\]

\[
\hat{D}(a, q) \hat{P} \hat{D}(a, q) = \hat{P} + q.
\]

It follows that when the displacement operator is applied to an arbitrary state \( | u \rangle \), it displaces the average transverse position (or, in quantum-mechanical language, the expectation value of the transverse coordinate operator) by the vector \( a \) and the average transverse momentum (or the expectation value of the transverse momentum operator) by the vector \( q \). We require that upon propagation through an optical system the displacement operator evolves in such a way that in the input plane \( z_0 \) and output plane \( z_1 \) of the optical system we have

\[
| v(z_0) \rangle = \hat{D}(a(z_0), q(z_0)) | u(z_0) \rangle,
\]

\[
| v(z_1) \rangle = \hat{D}(a(z_1), q(z_1)) | u(z_1) \rangle,
\]

where \( | v \rangle \) is the displaced state obtained when \( \hat{D} \) is applied to \( | u \rangle \). Since

\[
| v(z_1) \rangle = \hat{U} | v(z_0) \rangle, \quad | u(z_1) \rangle = \hat{U} | u(z_0) \rangle,
\]

it follows that the displacement operators in the input and output planes are related by

\[
\hat{D}(a(z_1), q(z_1)) = \hat{U} \hat{D}(a(z_0), q(z_0)) \hat{U}^\dagger.
\]

Since the optical elements are Gaussian, the displacement operator at the output plane of the optical system still has the general form (6), with different coefficients \( a \) and \( q \). In order to determine how \( a \) and \( q \) change when going from the input to the output plane of the optical system, we use the following properties:

\[
\hat{U}_f(L) \hat{R} \hat{U}_l(L) = \hat{R} - L / k \hat{P},
\]

\[
\hat{U}_f(f) \hat{P} \hat{U}_l(f) = \hat{P} + k / f \hat{R},
\]

where \( \hat{U}_l \) and \( \hat{U}_f \) are given in Eqs. (3) and (4), respectively. The values of \( a \) and \( q \) after propagation over a distance \( L \) in vacuum, for which we write \( a(L) \) and \( q(L) \), are expressed in terms of their initial values \( a(0) \) and \( q(0) \) by

\[
\begin{pmatrix}
  a(L) \\
  q(L) / k
\end{pmatrix} =
\begin{pmatrix}
  1 & L \\
  0 & 1
\end{pmatrix}
\begin{pmatrix}
  a(0) \\
  q(0) / k
\end{pmatrix}.
\]

For passage through a lens in the transverse plane \( z \), the values of \( a(z) \) and \( q(z) \) immediately after the lens are expressed in terms of their incoming values \( a(z) \) and \( q(z) \) by

\[
\begin{pmatrix}
  a(z) \\
  q(z) / k
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 \\
  -1 / f & 1
\end{pmatrix}
\begin{pmatrix}
  a(z) \\
  q(z) / k
\end{pmatrix}.
\]

We see that these matrices correspond to the \( ABCD \) matrices for the description of the change in position and
slope of a light ray propagating over a distance \(L\) and passing through a lens with focal distance \(f\), as discussed by Siegman.\(^7\) In the paraxial approximation the value \(q/k\), which is the transverse momentum divided by the total momentum, can indeed be interpreted as a slope. It follows that, upon propagation through the optical system, \(a\) and \(q/k\) transform in the same way as the position and slope of a light ray, respectively. For a general Gaussian optical system, the \(ABCD\) matrix is a \(4 \times 4\) matrix, but since the optical systems that we consider are cylindrical, it is sufficient to use \(2 \times 2\) matrices, since for the two components of the transverse vectors \(a\) and \(q\) the transformation is identical. For a Gaussian optical system, the \(ABCD\) matrix, which describes the propagation from the input plane \(z_0\) to the output plane \(z_1\), completely determines the corresponding evolution operator \(\hat{U}\) within a phase factor, and vice versa. In general we have
\[
\begin{pmatrix}
a(z_1) \\
q(z_1)/k
\end{pmatrix} = \begin{pmatrix} A & B \\
C & D \end{pmatrix} \begin{pmatrix} a(z_0) \\
q(z_0)/k
\end{pmatrix}. \tag{16}
\]
As a consequence, complete knowledge of a Gaussian optical system can be obtained by probing it with two light rays, for which either the position or the slope is zero in the input plane. Then the matrix elements of the \(ABCD\) matrix are determined by the positions and slopes of the two rays at the output plane of the optical system.

3. PROPAGATION OF GAUSSIAN BEAMS

Well-known solutions of the paraxial wave equation are the Hermite–Gaussian (HG) beams, which are also eigenmodes of a two-mirror resonator. They resemble the eigenfunctions of the two-dimensional (2D) quantum harmonic oscillator,\(^{11}\) which can be explained by using an operator method involving ladder operators, as we shall briefly discuss. We saw that upon propagation through a Gaussian optical system, the displacement operator retains the form (6), where the propagation is contained in the variation of the coefficients \(a\) and \(q\). The transformation rules (12) and (13) show that linear combinations of \(\hat{R}\) and \(\hat{P}\) remain linear combinations upon propagation through an optical system. The propagation is contained in coefficients \(\kappa\) and \(\beta\), which behave in a similar fashion as \(q\) and \(a\). These coefficients define a vector \(\hat{A}\) of two lowering operators \(\hat{a}_x\) and \(\hat{a}_y\) in an arbitrary transverse plane by
\[
\hat{A} = \frac{1}{\sqrt{2}} (\kappa \hat{R} + i \beta \hat{P}), \tag{17}
\]
where the coefficients \(\kappa\) and \(\beta\) are complex numbers. The corresponding vector \(\hat{A}^\dagger\) of raising operators \(\hat{a}_x^\dagger\) and \(\hat{a}_y^\dagger\) is given by
\[
\hat{A}^\dagger = \frac{1}{\sqrt{2}} (\kappa^* \hat{R} - i \beta^* \hat{P}). \tag{18}
\]
The ladder operators must satisfy the usual bosonic commutation rules:
\[
[\hat{a}_x, \hat{a}_y^\dagger] = [\hat{a}_y, \hat{a}_x^\dagger] = 1, \tag{19}
\]
These commutation rules are satisfied when
\[
\text{Re}(\kappa \beta^*) = 1. \tag{21}
\]
We require that the ladder operators transform solutions of the paraxial wave equation into other solutions. When the state \(|u(z_0)\rangle\) in the input plane gives the state \(|u(z_1)\rangle\) in the output plane, this implies that an input state \(\hat{A}(z_0)|u(z_0)\rangle\) leads to the output state \(\hat{A}(z_1)|u(z_1)\rangle\). This gives
\[
\hat{A}(z_1) = \hat{U}\hat{A}(z_0)\hat{U}^\dagger, \tag{22}
\]
which is identical to the transformation property of the displacement operator in Eq. (11). It follows that the coefficients \(\beta\) and \(i \kappa/k\) transform in the same way as \(a\) and \(q/k\) in Eq. (16), so that
\[
\begin{pmatrix} \beta(z_1) \\
i \kappa(z_1)/k
\end{pmatrix} = \begin{pmatrix} A & B \\
C & D \end{pmatrix} \begin{pmatrix} \beta(z_0) \\
i \kappa(z_0)/k
\end{pmatrix}. \tag{23}
\]
By using the ladder operators, we obtain a complete set of Gaussian beam profiles. In an arbitrary transverse plane, the lowest-order Gaussian beam profile \(u_{00}(R)\) \((R|u_{00})\) is defined by
\[
\hat{A}|u_{00}\rangle = 0. \tag{24}
\]
It follows that
\[
u_{00}(R) = \frac{1}{\beta \sqrt{\pi}} \exp\left(-\frac{\kappa R^2}{2 \beta^2}\right), \tag{25}
\]
which is normalized to unity, as can be checked by using Eq. (21). Because of the propagation property of \(\hat{A}\) in Eq. (22), this expression is valid in both the input and output planes of the optical system, as long as the values for \(\kappa\) and \(\beta\) that they acquire in the plane under consideration are taken.

We verify that in a region of free propagation, with the proper values of \(\kappa\) and \(\beta\), Eq. (25) satisfies the paraxial wave equation (2). From the \(ABCD\) matrix for free propagation, as specified in Eq. (14), it follows with Eq. (23) that \(\kappa\) is independent of \(z\) and that \(d \beta/dz = i \kappa/k\). This shows that the lowest-order Gaussian beam \(u_{00}(R,z)\) indeed satisfies the paraxial wave equation. It is customary to introduce for each transverse plane the complex parameter \(Q\), defined by\(^{7,9}\)
\[
\frac{1}{Q} = \frac{i \kappa}{\kappa} = \frac{1}{S} + i \frac{1}{k \gamma}, \tag{26}
\]
where \(S\) is the radius of curvature of the wavefront and \(\gamma\) is the spot size. It follows from Eq. (23) that the evolution of \(Q\) from the input plane \(z_0\) to the output plane \(z_1\) of the optical system is simply expressed by
\[
Q(z_1) = \frac{AQ(z_0) + B}{CQ(z_0) + D}, \tag{27}
\]
which is known as the \(ABCD\) law.\(^5\) Using Eq. (21), we find that \(\gamma = |\beta|\). Moreover, Eq. (26) demonstrates that the exponential in the fundamental mode profile (25) separates
into a curvature factor $\exp(ikR^2/2S)$ and a real Gaussian
$\exp(-R^2/2\gamma^2) = \exp(-R^2/2|\beta|^2)$.

We now wish to apply the raising operators to obtain the
higher-order Gaussian beam profiles. We use the fol-
lowing property:

$$\exp\left(\frac{ik}{2S}R^2\right)\hat{P}\exp\left(\frac{ik}{2S}R^2\right) = \hat{P} - k\hat{R},$$

where $S$ is the radius of curvature defined in Eq. (26).

This property can be read as $\hat{A}^\dagger = \sqrt{\frac{\beta}{\beta}} \exp\left(\frac{ik}{2S}R^2\right)\hat{B}^\dagger \exp\left(-\frac{ik}{2S}R^2\right)$, (29)

where $\hat{B}^\dagger$ is a vector of real raising operators, defined by

$$\hat{B}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{|\beta|}\hat{R} - i|\beta|\hat{P}\right).$$

This can be checked by using Eq. (26) and the relation $\gamma = |\beta|$. The higher-order Gaussian beam profiles are ob-
tained in the standard way by repeated application of the
raising operators expressed as in Eq. (29). We have

$$|u_{nm}(z_0) = \frac{1}{\sqrt{n!m!}}(\hat{a}^\dagger)^m(\hat{a})^n|u_{00}\rangle, \quad n, m = 0, 1, 2, \ldots.$$ (31)

The operator $\hat{B}^\dagger$ in Eq. (30) has the form that is familiar
from the quantum-mechanical description of the 2D har-
monic oscillator, and it produces the higher-order eigen-
functions when acting on the real Gaussian ground state
$\exp(-R^2/2|\beta|^2)$. The higher-order Gaussian beam profiles
$u_{nm}(R) = |R|^m|u_{nm}\rangle$ are the familiar HG beam profiles

$$u_{nm}(R) = \frac{1}{\sqrt{|\beta|^{n+m+1}\pi^{n+m}\pi}} H_n\left(\frac{\sqrt{\kappa}}{|\beta|}\right) H_m\left(\frac{\sqrt{\lambda}}{|\beta|}\right) \times \exp\left(-\frac{\kappa}{2\beta} R^2\right),$$ (32)

where

$$H_n(\xi) = \exp(\xi^2/2)\left(\xi - \frac{\partial}{\partial \xi}\right)^n \exp(-\xi^2/2), \quad n = 0, 1, 2, \ldots.$$ (33)

are the Hermite polynomials. Again, the propagation prop-
erty of $\hat{A}$ in Eq. (22) guarantees that, with the appro-
priate values of $\kappa$ and $\beta$, this expression is valid in both
the input and output planes of the optical system.

4. MODES OF AN OPTICAL RESONATOR

A. Resonance Condition

As we will demonstrate in this section, by using the oper-
ator method developed in Section 3, we can obtain the
frequency spectrum and the eigenmodes of a cylindrical
optical resonator. We consider an optical resonator of
length $L$, with end mirrors with focal distances $f_1$ and $f_2$,
located at $z=0$ and $z=L$. When we unfold the resonator
into an equivalent lens guide, we obtain the system dis-
played in Fig. 1, with length $2L$. The corresponding ev-
olution operator (5) represents the round-trip operator of
the resonator, with the input plane of mirror 1 as refer-
ence plane. An eigenmode of the resonator must repro-
duce itself after a round trip, which implies that the cor-
responding field in the input plane of the lens guide is an
eigenstate of the operator (5). When the transverse pro-
file of the field propagating to the right in the lens guide is
described by the function $u(R,z)$, with $0 \leq z \leq 2L$, then the
complex electric field in the resonator is

$$E(R,z,t) = E_0[u(R,z)\exp(ikz) - u(R,2L-z)$$

$$\times \exp(\pm2L(z-z))\exp(-i\omega t)]$$ (34)

for $0 \leq z \leq L$. The real electric field vanishes on the end
mirrors at $z=0$ and $z=L$.

The eigenstate of the operator (5) can be found from the
$ABCD$ matrix $M$ for the lens guide, which is identical to
the $ABCD$ matrix for the round trip in the resonator,
starting in the input plane of mirror 1. A resonator is
stable when the eigenvalues of $M$ have unit absolute values.\footnote{Since $M$ is a real matrix with unit determinant, one of the eigenvalues is then the complex conju-
gate of the other and one eigenvector is the complex conju-
gate of the other. The special case where the eigenvalues are identical, and therefore real, will be considered in Section 6. For complex eigenvectors of $M$ there exists no ray that transforms into itself after one round trip, since rays are described by the real position and slope. We saw in Section 3 that the evolution of the parameters $\kappa$ and $\beta$, which determine the ladder operators, is also governed by $M$, as expressed by Eq. (23). It is possible to find values $\kappa_0$ and $\beta_0$ for which

$$M\left(\begin{array}{c} \beta_0 \\ i\kappa_0/k \end{array}\right) = \exp(i\chi)\left(\begin{array}{c} \beta_0 \\ i\kappa_0/k \end{array}\right),$$ (35)

where $\exp(i\chi)$ is one of the eigenvalues of $M$ and where $\kappa_0$ and $\beta_0$ satisfy Eq. (21). Apart from a phase factor, this de-
fines in a unique way the lowering operator $\hat{A}_0$, which transforms into itself after a round trip.\footnote{By introducing

$$\hat{A}_0 = \frac{1}{\sqrt{2}}(\kappa_0 \hat{R} + i\beta_0 \hat{P}),$$ (36)

we find from Eq. (23) that

$$\hat{U}\hat{A}_0\hat{U}^\dagger = \exp(i\chi)\hat{A}_0,$$ (37)

where $\hat{U}$ is the evolution operator for the round trip. The cor-
responding raising operator $\hat{A}_0^\dagger$ is determined by the
other eigenvector of $M$. The Hermitian conjugate of Eq.
(37) expresses the round-trip evolution of $\hat{A}_0$. We see that
although the round-trip matrix $M$ has two eigenvectors,
there is only one Gaussian fundamental mode, since only
one of the eigenvectors can correspond to a lowering op-
erator.

With the values $\kappa_0$ and $\beta_0$, the HG profiles in Eq. (32),
which we shall designate by $u_{nm}(R;\kappa_0,\beta_0)$, transform into
themselves after a round trip up to a phase factor. It fol-
ows that $u_{nm}(R;\kappa_0,\beta_0)$ are the transverse profiles $u(R)$ of
the eigenmodes of the resonator in the input plane of mirror 1. To find the profiles in a different transverse plane, we must obtain the $ABCD$ matrix for the propagation in the allowed direction to this plane. Then the values that $\kappa$ and $\beta$ acquire in that plane are obtained from Eq. (23) and the mode profiles are found by using Eq. (32).

After a round trip, $\beta_0$ picks up a phase factor $\exp(i\chi)$. We see then from Eq. (25) that after a round trip the fundamental mode profile acquires a phase factor $\exp(-i\chi)$, which is the round-trip Gouy phase. This Gouy phase is completely determined by the characteristics of the resonator, since it is an eigenvalue of the round-trip $ABCD$ matrix. It follows from Eq. (32) that

$$\hat{U}|u_{nm}(\kappa_0, \beta_0)\rangle = \exp[-i(n+m+1)\chi]|u_{nm}(\kappa_0, \beta_0)\rangle.$$  

Without loss of generality we have left out the overall phase of $\hat{U}$ in this expression. Taking into account the phase due to the plane-wave part of the electric field in Eq. (1), we obtain the well-known resonance condition:

$$2kL = (n+m+1)\chi + 2\pi l,$$  

(39)

where $l$ is an integer and $2L$ is the length of the lens guide. The integer $l$ has the significance of the longitudinal mode number, which determines the number $l+1$ of transverse nodal planes.

B. Degeneracy

It is clear from Eq. (39) that the eigenmodes with mode numbers $n$ and $m$ for which $n+m$ attains the same value are degenerate. Any linear combination of eigenmodes with the same value of $n+m$ is also an eigenmode of the resonator. These can be Laguerre–Gaussian (LG) modes or modes that are between LG and HG modes.\textsuperscript{13–16}

When the Gouy phase takes the value

$$N\chi = 2\pi K,$$  

(40)

where $N$ and $K$ are integers, it follows from Eq. (39) that eigenmodes with mode numbers $n$ and $m$ for which $n+m$ differs by a multiple of $N$ are degenerate. When $n+m$ is increased by $N$ and $l$ is decreased by $K$, the resonance condition is satisfied for the same wavenumber $k$. Without loss of generality we can assume that $N$ and $K$ have no common divisor. In that case there are no other modes with the same wavenumber. When the condition of degeneracy (40) is satisfied, we have $\hat{M}^N = 1$ and, equivalently, $\hat{U}^N = 1$, where a possible overall phase of $\hat{U}$ is left out for simplicity. Then the eigenvalues of the unitary operator $\hat{U}$ belong to the finite set of $N$ values $\exp(-2\pi is/N)$, with $s = 0, 1, \ldots, N-1$.

It is illuminating to identify $N$ projection operators $\hat{V}_s$ on the subspaces of transverse eigenmodes with eigenvalue $\exp(-2\pi is/N)$. These operators can be expressed as

$$\hat{V}_s = \frac{1}{N} \sum_{r=0}^{N-1} \exp\left(\frac{2\pi irs}{N}\right) \hat{U}^r, \quad s = 0, 1, \ldots, N-1.$$  

(41)

From direct substitution it follows that

$$\hat{U}\hat{V}_s = \exp\left(-\frac{2\pi is}{N}\right)\hat{V}_s,$$  

(42)

which shows that for an arbitrary state $|u\rangle$ the state $\hat{V}_s|u\rangle$ is an eigenstate of the round-trip operator $\hat{U}$, with eigenvalue $\exp(-2\pi is/N)$. Moreover, the set of operators $\hat{V}_s$, with $s = 0, 1, \ldots, N-1$ is complete, in the sense that they add up to the unit operator. From Eqs. (38) and (40) one checks that $\hat{V}_s|u_{nm}(\kappa_0, \beta_0)\rangle = |u_{nm}(\kappa_0, \beta_0)\rangle$, provided that $\exp[-2\pi i(n+m+1)K/N] = \exp(-2\pi is/N)$. This implies that

$$K(n+m+1) = pN + s,$$  

(43)

with $p$ an integer. The corresponding values of $n+m$ differ by a multiple of $N$ and define modes with wavenumbers specified by the resonance condition

$$2kL = 2\pi l' + 2\pi s/N,$$  

(44)

with $l' = l+p$. The resonant wavenumbers (44) form an equidistant mesh with separation determined by $L\Delta k = \pi/N$, which is $1/N$ times the free spectral range of the resonator with length $L$.

5. GEOMETRIC MODES

A. Geometric Picture

The structure of the projection operators (41) can be understood directly from a geometric picture of modes in the resonator. We consider the lens guide in Fig. 2, which consists of a sequence of $N$ times the unfolded resonator of Fig. 1 and thus stretches from $z = 0$ to $z = 2NL$. In the case of degeneracy the round-trip $ABCD$ matrix for the resonator satisfies $\hat{M}^N = 1$. It follows that for the $N$-fold lens guide the $ABCD$ matrix is the unit matrix and also that the unitary operator that describes the propagation from the plane $z = 0$ to the plane $z = 2NL$ is the unit operator. Therefore any transverse beam profile $u(R,0)$ at the plane $z = 0$ transforms into itself after propagation to the plane $z = 2NL$. We write $u(R,z)$ for the $z$-dependent profile, where $0 \leq z \leq 2NL$. It follows that $u(R,2NL) = u(R,0)$. The same periodicity holds for the traveling wave $\exp(ikz)u(R,z)$, provided that the wavenumber $k$ obeys the requirement

$$2kNL = 2\pi s',$$  

(45)

with $s'$ an integer. When this resonance condition holds, the $N$-fold lens guide can be folded into the resonator of length $L$, with the complex electric field
\[ E(R,z,t) = E_0 \sum_{p=0}^{N-1} u(R,2pL + z) \exp[\imath k(2pL + z)] \times \exp(-\imath \omega t) - E_0 \sum_{p=0}^{N-1} u(R,2(p+1)L - z) \times \exp[\imath k(2(p+1)L - z)] \exp(-\imath \omega t), \] 

(46)

where now \( 0 \leq z \leq L \). The terms on the first and second line describe the field propagating to the right and the left, respectively. When we compare the requirement (45) with the resonance condition for degeneracy in Eq. (44), we see that \( s' = s + N' \). When a value of \( s' \) is chosen, the value of \( s \), which determines the round-trip Gouy phase by Eq. (42), is given by \( s' \mod N \). Note that the transverse profile \( u(R,0) \) in the input plane of the lens guide can be freely chosen, within the validity range of the paraxial approximation. The field in the reference plane of the resonator, i.e., the input plane of mirror 1, is specified by the second line in Eq. (46), with \( z = 0 \). The resulting expression is equivalent to the action of the projection operator \( \hat{V} \), as defined by Eq. (41), on the input field. This proves that the total field in the reference plane of the resonator is an eigenstate of the round-trip operator \( \hat{U} \), with eigenvalue \( \exp(-2\pi i s/N) \).

B. Displaced State

We consider the evolution of the displacement operator in the \( N \)-fold lens guide in Fig. 2. As discussed in Section 2, the displacement operator, as defined in Eq. (6), shifts the average transverse position and momentum by \( a \) and \( q \), respectively. During propagation through the lens guide, \( a \) and \( q/k \) transform in the same way as the position and slope of a ray. We write \( a(z) \) and \( q(z) \) for their values in the plane \( z \). If for an arbitrary state \( |v(z)\rangle \) the average position and momentum vanish in a plane \( z_0 \), that is, if \( \langle v(z) | \hat{R} | v(z_0) \rangle = 0 \) and \( \langle v(z) | \hat{P} | v(z_0) \rangle = 0 \), then the average transverse position and momentum vanish in any other transverse plane as well. This is because, according to Eqs. (12) and (13), the transformation rules of \( \hat{R} \) and \( \hat{P} \) are linear and homogenous. It follows that the center of the beam described by the state

\[ |u(z)\rangle = \hat{D}(a(z),q(z))|v(z)\rangle \] 

(47)

follows the trajectory of a ray inside the \( N \)-fold lens guide. In the case of \( N \)-fold degeneracy, we have \( |v(2NL)\rangle = |v(0)\rangle \), \( a(2NL) = a(0) \), and \( q(2NL) = q(0) \). We fold the \( N \)-fold lens guide into a resonator. Then the electric field inside the resonator is described by Eq. (46). The electric field is a linear combination of \( 2N \) displaced beams, of which \( N \) propagate to the right and \( N \) to the left. When following one of the displaced beams during \( N \) round trips, the displaced beam transforms into another displaced beam after each round trip, finally to transform into itself after \( N \) round trips. The center of the displaced beam follows a closed trajectory, which means that the mode is a geometric mode.

C. Electric Field of Geometric Modes

In order to find an expression for the electric field of a geometric mode, it is necessary to obtain an expression for the transverse profile of a displaced state. In an arbitrary plane the displaced state \( |u\rangle \) is defined in terms of the arbitrary state \( |v\rangle \) by

\[ |u\rangle = \hat{D}(a,q)|v\rangle. \]

(48)

In the case in which the commutator of two operators \( \hat{A} \) and \( \hat{B} \) commutes with both \( \hat{A} \) and \( \hat{B} \), the following relation holds:

\[ \exp(\hat{A} + \hat{B}) = \exp(\frac{-1}{2}[\hat{A},\hat{B}])\exp(\hat{A})\exp(\hat{B}). \]

(49)

We use this to express the displacement operator in Eq. (6) as

\[ \hat{D}(a,q) = \exp(-i\hat{q}a/2)\exp(i\hat{q}\hat{R})\exp(-ia\hat{P}). \]

(50)

Using this expression, we find that

\[ \langle R'|\hat{D}|R\rangle = \exp(-i\hat{q}a/2)\exp(i\hat{q}\hat{R})\delta(R - R' - a). \]

(51)

It follows that

\[ u(R) = \exp(-i\hat{q}a/2)\exp(i\hat{q}\hat{R})v(R - a). \]

(52)

Equation (52) expresses the transverse profile \( u(R) \) of the displaced state \( |u\rangle \) in terms of the profile \( v(R) \) of the state \(|v\rangle\).

As an example we consider geometric modes consisting of displaced beams that are obtained by displacing the Gaussian fundamental mode of the resonator \( u_{00}(R; \kappa_0, \beta_0) \). Then the displaced beams all have the focus in the middle of the resonator. The energy density of a mode is calculated by averaging the square of the real part of the complex electric field over time and over a range of \( z \) of several times the wavelength that is still small enough for the beam profile to be considered constant. Then the energy density is simply the sum of the squared absolute values of the right- and left-propagating parts of the beam profile. It follows that only beams that propagate in the same direction can give rise to interference fringes in the energy density profile. The energy density profiles of two geometric modes are depicted in Fig. 3 for three-fold degeneracy. The focal distances of the mirrors are equal to the length of the resonator. The values \( k f_1 = k f_2 = k L = 3 \pi \times 10^4 \) are used, which correspond to a zero round-trip Gouy phase, as follows from Eq. (45) and the discussion below. In the vertical, or transverse, direction, the pictures are magnified by a factor of 4 compared with the horizontal direction, which is parallel to the optical axis. Interference fringes occur at the crossings of beams that propagate in the same direction.

Our expressions for the displaced beams differ from those used by Chen et al.16 Inspired by the expressions for spin-coherent states, these authors use a finite expansion of HG modes with indices that differ by multiples of the degeneracy number. The weights are binomial and contain arbitrarily chosen parameters. We find expressions for the eigenmodes in which the significance of the parameters is fully specified by the requirement that the beams be displaced Gaussian fundamental modes.
6. SPECIAL LIMITING CASES

In the section we consider resonators for which the eigenvalues of the round-trip $ABCD$ matrix $M$ are $\pm 1$, so that the Gouy phase $\chi$ is a multiple of $\pi$. Since $M$ has unit determinant, the eigenvalues must be identical. In order to find the ladder operators that determine the eigenmodes of the resonator, we must find values of $\kappa_0$ and $\beta_0$ that satisfy Eq. (35). When $M$ is $\pm 1$ times the unit matrix, Eq. (35) is satisfied for all values of $\kappa_0$ and $\beta_0$. Still, not all values are allowed, since the requirement in Eq. (21) must be fulfilled, which follows from the commutation rules for the ladder operators. When $M$ is not $\pm 1$ times the unit matrix, it has only one eigenvector. Since $M$ is a real matrix, this eigenvector must be real. For the eigenvector in Eq. (35) to be real, $\kappa_0$ must be purely imaginary and $\beta_0$ must be real. It follows that no ladder operators can be defined, since the requirement $\text{Re}(\kappa_0\beta_0)=1$ in Eq. (21) cannot be satisfied. Therefore, when $M$ has real eigenvalues and is not equal to $\pm 1$ times the unit matrix, the resonator has no Gaussian eigenmodes. Also, there exists no value of $N$ for which $M^N=1$. Still, the resonator has one eigenray.

These cases can be illustrated for a symmetric resonator. Then it is sufficient to consider the $ABCD$ matrix for only half the round trip, which corresponds to propagation from the plane $z=0$ to the plane $z=L$ in the lens guide of Fig. 1. We write $f$ for the focal distance of both mirrors. The $ABCD$ matrix $M_h$ for half the round trip is found by multiplying the lens matrix in Eq. (15) on the left by the matrix for free propagation in Eq. (14). We find that

\[ M_h(g) = \begin{bmatrix} 2g - 1 & L \\ -2(1-g)/L & 1 \end{bmatrix}, \tag{53} \]

where $g = 1 - L/2f$ is the $g$ parameter of the resonator. For stable resonators the eigenvalues $g \pm i\sqrt{1-g^2}$ of $M_h(g)$ must have unit absolute values, which means that $-1 \leq g \leq 1$. The eigenvalues of the full round-trip $ABCD$ matrix $M = M_h^2(g)$ are real when $g$ takes the values $1$, $-1$, and $0$.

When $g = 1$, both mirrors of the resonator are flat and the full round-trip $ABCD$ matrix $M = M_h^2(1)$ is given by

\[ M = \begin{bmatrix} 1 & 2L \\ 0 & 1 \end{bmatrix}. \tag{54} \]

The real vector $(1,0)$ is the only eigenvector of this matrix, with eigenvalue 1, and evidently corresponds to the only eigenray of this resonator, which is a horizontal ray with no slope. As discussed above in the first paragraph of this section, this resonator does not sustain Gaussian modes, since the eigenvector is real. It follows then that there are also no geometric modes possible. For the round-trip $ABCD$ matrix $M$ in Eq. (54), there is indeed no value of $N>0$ for which $M^N=1$.

In a concentric resonator, for which $g = -1$, the surfaces of the spherical mirrors lie on the surface of a sphere. The full round-trip $ABCD$ matrix $M = M_h^2(-1)$ is given by

\[ M = \begin{bmatrix} 5 & -2L \\ 8/L & -3 \end{bmatrix}, \tag{55} \]

for which $(L/2,1)$ is the only eigenvector, with eigenvalue 1. The corresponding eigenray originates from the center of the sphere on which the mirror surfaces lie. Again, no appropriate values of $\kappa_0$ and $\beta_0$ exist, so that a concentric resonator also does not have Gaussian eigenmodes and hence has no geometric modes.

In a symmetric confocal resonator, for which $g = 0$, the focal points of the mirror coincide, and the full round-trip $ABCD$ matrix is $M = M_h^2(0) = -1$. Every ray is then an eigenray for a single round trip, with eigenvalue $-1$. Also, all values of $\kappa_0$ and $\beta_0$ satisfying Eq. (21) can be used to
define ladder operators. As a consequence, every Gaussian beam is an eigenmode of the symmetric confocal resonator. This means that the focal planes of the right- and left-propagating beams do not have to coincide and that, when they do, the focus does not have to be in the middle of the resonator. Because of the twofold degeneracy, every transverse profile transforms into itself after two round trips, which means that the symmetric confocal resonator does have geometric modes. It follows from Eq. (38) that the two eigenspaces of the round-trip evolution operator for eigenvalues ±1 are spanned by the HG eigenstates for which \( n + m + 1 \) is even and odd, respectively. 12

7. CONCLUSIONS
We used an operator method to study the modes of a resonator. The Gaussian eigenmodes of the resonator were generated by introducing ladder operators. The round-trip Gouy phase of the resonator, which is completely determined by the geometry of the resonator, and therefore independent of the wavelength of the light, determines the properties of the Gaussian eigenmode. The displacement operator was used to create displaced beams, with a center that follows the trajectory of a ray. For the case of \( N \)-fold degeneracy, \( N \) projection operators were defined that project onto modes of the resonator, each with a different Gouy phase. The geometric argument to obtain modes of the resonator in the case of degeneracy consists of folding a light beam propagating through a sequence of \( N \) times the lens guide of the unfolded resonator back into a resonator. This geometric argument was shown to be equivalent to the operation of one of the projection operators. The beam parameters, such as shape and size of spot and curvature, can be freely chosen in one transverse plane. In the case of degeneracy, when a displaced beam in the sequence of lens guides is folded into a resonator, it gives rise to a geometric mode, which consists of beams that follow a closed trajectory.

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