Angular spectrum of quantized light beams

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Received October 10, 2005; accepted October 30, 2005; posted November 9, 2005 (Doc. ID 65286)

We introduce a generalized angular spectrum representation for quantized light beams. By using our formalism, we are able to derive simple expressions for the electromagnetic vector potential operator in the case of (a) time-independent paraxial fields, (b) time-dependent paraxial fields, and (c) nonparaxial fields. For the first case the well-known paraxial results are fully recovered. © 2006 Optical Society of America

OCIS codes: 000.1600, 270.0270.

Propagation of nonclassical states of the electromagnetic field is an issue of growing interest in quantum optics for both fundamental and technological purposes. Consider, for instance, the relevance of propagation of entangled photons to quantum cryptographic systems.

Our purpose in the present Letter is to provide a novel perfectly general formalism for the representation of quantized light beams that can be used in any regime of propagation. Such an objective is achieved by using a dispersion relation that some of us have recently introduced in Ref. 3 and a generalized angular spectrum representation for the field operators. The usefulness of our approach becomes apparent whenever one deals with quantum systems for which both paraxial and nonparaxial regimes of propagation may be relevant such as, e.g., downconverted photon pairs.

Consider the plane-wave expansion of the positive-frequency part of the electromagnetic vector potential operator \( \hat{A}(r,t) = \hat{A}^+(r,t) + \text{H.c.} \) in the Coulomb gauge

\[
\hat{A}^+(r,t) = \int d^3k \left( \frac{\hbar}{16\pi^2\varepsilon_0|k|} \right)^{1/2} \times 2 \sum_{\lambda=1}^{\infty} e^{i\lambda/k_0} \hat{a}_\lambda(k) \exp(i\mathbf{k} \cdot \mathbf{r} - ic|k|t).
\]

Since we want to describe fields propagating mainly along the z axis, we find it convenient to define \( k_z = s\xi \), where \( s = \text{sign}(k_z) = \pm 1 \), and \( \xi \gg 0 \). Then we can rewrite Eq. (1) as

\[
\hat{A}^+(r,t) = \sum_{s=\pm 1} \int dk_z dk_0 \int_0^{\infty} d\xi \left( \frac{\hbar}{16\pi^2\varepsilon_0|k_z|} \right)^{1/2} \times 2 \sum_{\lambda=1}^{\infty} e^{i\lambda/k_0} \hat{a}_\lambda(k_z) \exp(i\mathbf{k}_z \cdot \mathbf{r} - ic|k_z|t),
\]

where we have defined \( \mathbf{k}_z = (k_x, k_y, s\xi) \). The field annihilation and creation operators satisfy the canonical commutation relations

\[
[\hat{a}_\lambda(k_z), \hat{a}^\dagger_{\lambda'}(k_z')] = \delta_{\lambda\lambda'} \delta_{\sigma\sigma'} (q - q') \delta(\xi - \xi').
\]

Let \( \omega > 0 \) be an arbitrary frequency; at a later point in this Letter we shall identify \( \omega \) with the carrier frequency of a paraxial field. We perform a change of variables \( \{k_x, k_y, \xi\} \to \{q_x, q_y, \omega\} \) such that

\[
k_x = q_x, \quad k_y = q_y, \quad \xi = f(q_x, \omega),
\]

where \( f(q_x, \omega) \geq 0 \) is an (almost) arbitrary function to be determined. For reasons that will soon be clear, we require \( f(q_x, \omega) \) to increase monotonically for increasing \( \omega \) in the domain

\[
\mathcal{I}_\omega(f, q) = \{ \omega \in \mathbb{R}^+: f(q_x, \omega) \geq 0 \}.
\]

This condition implies that \( df(q_x, \omega)/d\omega > 0 \) for \( \omega \in \mathcal{I}_\omega(f, q) \). Therefore, in such a domain, we can write

\[
\delta(\mathbf{q} - \mathbf{q'}) \delta(\xi - \xi') = \delta(\mathbf{q} - \mathbf{q'}) \delta(\omega - \omega').
\]

If we substitute Eq. (6) into Eq. (3), we obtain

\[
[\hat{a}_\alpha(q_x, q_y, \omega), \hat{a}^\dagger_{\alpha'}(q_x', q_y', \omega')] = \delta_{\alpha\alpha'} \delta_{\sigma\sigma'} \delta(\omega - \omega').
\]

Equation (7) suggests the introduction of the angular-spectrum field operators \( \hat{\alpha}_{\lambda\lambda'}(q_x, q_y, \omega), \) defined as

\[
\hat{\alpha}_{\lambda\lambda'}(q_x, q_y, \omega) = \hat{a}_{\lambda}(q_x, q_y, f(q_x, \omega))) \frac{df(q_x, \omega)}{d\omega},
\]

By using Eqs. (8) and (7) it is easy to see that

\[
[\hat{\alpha}_{\lambda\lambda'}(q_x, q_y, \omega), \hat{\alpha}^\dagger_{\lambda\lambda'}(q_x', q_y', \omega')] = \delta_{\lambda\lambda'} \delta_{\sigma\sigma'} \delta(\omega - \omega').
\]

Equation (9) is the first main result of this Letter; it worthwhile to note that it is exact. No approximations were made to obtain it.

The condition \( f(q_x, \omega) \geq 0 \) defines a volume \( V_{q_x, \omega}(f) \) in the half-space \( \mathbb{R}^+ \times \mathbb{R}^+ \) spanned by \( \{q_x, q_y, \omega\} \). This volume is bounded by the surface \( S_{q_x, \omega}(f) = \partial V_{q_x, \omega}(f) \) defined by the equation \( f(q_x, \omega) = 0 \). If we define the two-dimensional domain \( \mathcal{C}_\omega(f, \mathbf{q}) = \{(q_x, q_y) \in \mathbb{R}^2: f(q_x, \omega) \geq 0 \} \), then from Eq. (5) it readily follows that

\[
\int_{\mathbb{R}^2} d^2q \int_{\mathcal{I}_\omega(f, \mathbf{q})} d\omega = \int_{\mathbb{R}^+} d\omega \int_{\mathcal{C}_\omega(f, \omega)} d^2q.
\]

We use this equality to rewrite Eq. (2) immediately in the new variables \( \{q_x, q_y, \omega\} \) as

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$$\hat{A}^{(+)}(r,t) = \sum_{s=\pm 1} \int_0^\infty d\omega \int_{c_q(f,\omega)}^2 d^2q \left( \frac{\hbar}{16\pi^2\epsilon_0 c \sqrt{q^2 + f^2}} \right)^{1/2} \times \sum_{\lambda=1}^2 e^{i(\lambda)(q,s)f} \hat{a}_{\lambda,s}(q,\omega) \hat{a}_{\lambda,s}^\dagger(q,\omega) \times \exp(iq \cdot x + isfz - itc^{-1}(q^2 + f^2)), \quad (11)$$

where \( q^2 + q_{\gamma}^2 + f = f(q,\omega), \) and \( x = (x,y) \). Note that the square root of the Jacobian \( \frac{df}{d\omega} \) was used to pass from the original operators \( \hat{a}_s(k_s) \) to the angular-spectrum operators \( \hat{a}_{\lambda,s}(q,\omega) \). Since we want to develop a formalism suitable for both nonparaxial and paraxial light beams, we rewrite Eq. (11) as

$$\hat{A}^{(+)}(r,t) = \sum_{s=\pm 1} \int_0^\infty d\omega \exp[-i\omega(t-sz/c)] \hat{\psi}_s(r,t), \quad (12)$$

so that \( \omega \) determines the plane carrier wave and \( \hat{\psi}_s(r,t) \) is the envelope field, which, at this stage, is not required to be spatially and temporally slowly varying:

$$\hat{\psi}_s(r,t;\omega) = \int_{c_q(f,\omega)}^2 d^2q \left( \frac{\hbar}{16\pi^2\epsilon_0 c \sqrt{q^2 + f^2}} \right)^{1/2} \times \sum_{\lambda=1}^2 e^{i(\lambda)(q,\omega)f} \hat{a}_{\lambda,s}(q,\omega) \times \exp[iq \cdot x + isfz(\omega/c)] \times \exp[-itc^{-1}(q^2 + f^2) - \omega]. \quad (13)$$

Moreover, since \( k_s = \hat{k}_s/\sqrt{q^2 + f^2} \), where \( \hat{k}_s = (q\hat{q} + sf\hat{z})/\sqrt{q^2 + f^2} \), and \( q\hat{q} = q(q,\omega) \), we have defined \( e^{(\lambda)}(q,\omega) = \hat{e}_s(q,\omega) = s \hat{z} \times \hat{q} \) and

$$e^{(1)}(q,\omega) = (\hat{q} \cdot s\hat{z})q/\sqrt{q^2 + f^2}. \quad (14)$$

Until now we furnished expressions for the field operators in the angular-spectrum representation, but not, for the energy, the momentum, etc. However, closed expressions for these physical quantities can be easily found by noting that the product

$$\hat{a}_{\lambda}(k_s) \hat{a}_{\lambda}^\dagger(k_s) d\zeta = \hat{a}_{\lambda,s}(q,\omega) \hat{a}_{\lambda,s}^\dagger(q,\omega) d\omega \quad (15)$$

is invariant with respect to the change of variables, Eq. (4). Then, for example, starting from the well-known expression for the Hamiltonian operator of the electromagnetic field (see, e.g., Ref. 7), after a straightforward calculation one obtains

$$\hat{H} = \frac{1}{2} \sum_{s=\pm 1} \int_0^\infty d\omega \int_{c_q(f,\omega)}^2 d^2q hc \sqrt{q^2 + f^2} \left( \frac{\hbar}{16\pi^2\epsilon_0 c \sqrt{1 + \vartheta^2}} \right)^{1/2} \times \sum_{\lambda=1}^2 \left[ \hat{a}_{\lambda,s}^\dagger(q,\omega) \hat{a}_{\lambda,s}(q,\omega) + \hat{a}_{\lambda,s}(q,\omega) \hat{a}_{\lambda,s}^\dagger(q,\omega) \right]. \quad (16)$$

Similar calculations can be easily done for the other relevant quantities. Equation (16) shows that, as expected for an arbitrary field, the frequency \( \omega \) of the carrier plane wave is not equal to the frequency \( c|\mathbf{k}| = c\sqrt{q^2 + f^2}(q,\omega) \) of the plane-wave mode \( \exp(i\mathbf{k} \cdot \mathbf{r}) \). However, as we shall see below, \( c|\mathbf{k}| \) reduces to \( \omega \) in the paraxial limit.

At this point the function \( f(q,\omega) \) is still undetermined; therefore we can exploit this freedom by imposing some constraints on the envelope field \( \hat{\psi}_s(r,t;\omega) \), which is, until now, perfectly general. In particular, we want to find an expression for the envelope field in which the Fresnel propagator\(^6\) plays a role even beyond the paraxial regime. To this end, we proceed as in Ref. 3, and we require \( \hat{\psi}_s(r,t=0;\omega) = \hat{\psi}_s(r;\omega) \) to satisfy the time-independent paraxial equation:

$$\frac{\partial^2 \hat{\psi}_s(r;\omega)}{\partial x^2} + \frac{\partial^2 \hat{\psi}_s(r;\omega)}{\partial y^2} + 2is/c \frac{\partial \hat{\psi}_s(r;\omega)}{\partial z} = 0. \quad (17)$$

In this way we obtain an expression for \( \hat{A}^{(+)}(r,t) \) that is an exact solution of the full d’Alembert equation for any time \( t>0 \), and its corresponding envelope field \( \hat{\psi}_s(r,t=0;\omega) \) satisfies the time-independent paraxial wave equation at \( t=0 \), as the initial condition. If we substitute from Eq. (13) the plane-wave \( \exp[iq \cdot x + izs(f - \omega/c)] \) into Eq. (17), we easily find

$$f(q,\omega) = \frac{\omega}{c} \left( 1 - \frac{q^2 c^2}{2\omega^2} \right). \quad (18)$$

For \( q \in \mathbb{C}_q(f,\omega) \) this function satisfies all our requirements: it is positive and \( df(q,\omega)/d\omega = (1 + \vartheta^2)/\omega > 0 \). We have defined \( \vartheta = qc/(\sqrt{2}\omega) \). It is easy to see that the plane-wave frequency \( c|\mathbf{k}| = \omega(1 + \vartheta^4)^{1/2} \) reduces to \( \omega \) in the paraxial limit \( \vartheta \ll 1 \). Finally, a closed expression for the field operator \( \hat{A}^{(+)}(r,t) \) can be given as

$$\hat{A}^{(+)}(r,t) = \sum_{s=\pm 1} \int_0^\infty d\omega \exp[-i\omega(t-sz/c)] \times \int_{c_q(f,\omega)}^2 d^2q \left( \frac{\hbar(1 + \vartheta^2)}{16\pi^2\epsilon_0 c \sqrt{1 + \vartheta^2}} \right)^{1/2} \times \sum_{\lambda=1}^2 e^{(\lambda)}(q,\omega) \hat{a}_{\lambda,s}(q,\omega) \exp\left( iq \cdot x - is\frac{q^2 c}{2\omega} \right) \times \exp[-i\omega t/(1 + \vartheta^2 - 1)]. \quad (19)$$

Equation (19) is the second main result of this Letter. It is easy to recognize, in the exponential function in the third row, the sought Fresnel propagator in momentum space. The spatial behavior of the envelope field is entirely governed by this term. It worth noting that Eq. (19) is exact, that is, it has been obtained without any approximation, and therefore it holds for both nonparaxial (\( \vartheta \ll 1 \)) and paraxial (\( \vartheta \ll 1 \)) beams. In the latter case the slowly varying term \( \exp[-i\omega t/(1 + \vartheta^2 - 1)] \) shows that the envelope field
\(\hat{\Psi}_s(\mathbf{r}, t; \omega)\) cannot be strictly monochromatic for any \(t > 0\).

In the remaining part of this Letter we give two different examples of the application of our theory in order to illustrate its generality. As a first example, let us generalize the previous case and require \(\hat{\Psi}_s(\mathbf{r}, t; \omega)\) to satisfy the time-dependent paraxial wave equation, for any \(t\):

\[
\frac{\partial^2 \hat{\Psi}_s}{\partial x^2} + \frac{\partial^2 \hat{\Psi}_s}{\partial y^2} + 2i \frac{c}{\omega} \frac{\partial \hat{\Psi}_s}{\partial z} = 0,
\]

where \(\hat{\Psi}_s = \hat{\Psi}_s(\mathbf{r}, t; \omega)\) for short. If we substitute from Eq. (13) the relevant term \(\exp[i \mathbf{q} \cdot \mathbf{x} + i s(\mathbf{r} - \omega/c)]\) into Eq. (20), we obtain a new dispersion relation

\[
f(\mathbf{q}, \omega) = \frac{\omega}{c} \left(1 - \frac{q^2 c^2}{4 \omega^2}\right).
\]

Once again, for \(\mathbf{q} \in C_q(f, \omega)\) this function satisfies all our requirements: it is positive and \(d f(\mathbf{q}, \omega)/d \omega = 1 + \eta^2/c > 0\), where \(\eta = q c/(2 \omega)\). It is easy to see that the plane-wave frequency \(c|\mathbf{k}|\) becomes \(c|\mathbf{k}| = \omega(1 + \eta^2)\). Also, for this case a closed expression for the field operator \(\hat{\mathbf{A}}(\mathbf{r}, t)\) can be given:

\[
\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{s=1}^{\infty} \int_0^d d \omega \exp[-i \omega(t - sz/c)]
\times \left(\frac{h}{16 \pi^3 \epsilon_0 c \omega}\right)^{1/2}
\times \sum_{l=1}^{2} \epsilon_l(\mathbf{q}, \omega) \hat{G}_{ls}(\mathbf{q}, \omega)
\times \exp\left[i \mathbf{q} \cdot \mathbf{x} - i \frac{q^2 c^2}{4 \omega}(sz + ct)\right].
\]

This expression is quite simpler than Eq. (19). However, its exponential part (in the last row) differ is by a factor of 1/2 from the Fresnel propagator expression. Once again, we stress that Eq. (22) is exact; no approximations were made.

As a last example of the application of our formalism, we choose to determine the function \(f(\mathbf{q}, \omega)\) by requiring \(\omega\) to coincide with the plane-wave frequency \(c|\mathbf{k}|: \omega = c|\mathbf{k}|\). It is easy to see that in this case we have

\[
f(\mathbf{q}, \omega) = \frac{\omega}{c} \left(1 - \frac{q^2 c^2}{\omega^2}\right)^{1/2} = \frac{\omega}{c} \left(1 - \frac{q^2 c^2}{2 \omega^2}\right),
\]

where the last approximate equality holds in the paraxial limit \(qc/\omega \ll 1\). For \(\mathbf{q} \in C_q(f, \omega)\) this function is positive and \(d f(\mathbf{q}, \omega)/d \omega = 1/(c\sqrt{1 - (qc/\omega)^2}) > 0\); therefore all our requirements are fulfilled. As expected, in the paraxial limit Eq. (23) coincides with Eq. (18). Since by definition \(\zeta = |\hat{k}_s| = (\omega/c) \cos \theta_i\), it follows that \(\cos \theta_i = f(\mathbf{q}, \omega) c/\omega\), and we can write

\[
\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_{s=1}^{\infty} \int_0^d d \omega \exp[-i \omega(t - sz/c)]
\times \left(\frac{h}{16 \pi^3 \epsilon_0 c \omega}\right)^{1/2}
\times \sum_{l=1}^{2} \epsilon_l(\mathbf{q}, \omega) \hat{G}_{ls}(\mathbf{q}, \omega)
\times \exp\left[i \mathbf{q} \cdot \mathbf{x} - i \frac{q^2 c^2}{4 \omega}(sz + ct)\right].
\]

Equation (24) is our last result. It gives an exact expression for the electromagnetic potential vector operator of a generic, nonparaxial light beam, in the angular-spectrum representation. By expanding in Taylor series the \(\cos \theta_i\) term around \(\theta = 0\), it is easy to see that Eq. (24) reduces to the classical paraxial expression (with the quantum operators \(\hat{a}_{ls}(\mathbf{q}, \omega)\) replaced by the corresponding classical amplitudes). Moreover, at the lowest order in \(\theta_i\), it coincides with Eq. (19) calculated at the lowest order in \(\theta_i\).

In conclusion, in this Letter we have presented a novel formalism for the representation of arbitrary quantized light beams. First we introduced an angular-spectrum representation for the field annihilation and creation operators. Then we used our formalism to derive an exact expression for the paraxial-like envelope field of a light beam. Finally, we illustrated the generality of our theory by applying it to the description of time-dependent, paraxial and nonparaxial light beams. Note that, although our formalism is fully quantum, all the previous results can be straightforwardly extended to classical fields just by replacing the quantum operators \(\hat{a}_{ls}(\mathbf{q}, \omega)\) with the corresponding classical amplitudes \(a_{ls}(\mathbf{q}, \omega)\).

We acknowledge support from the EU under the IST-ATESIT contract. This project is also supported by FOM. A. Aiello’s e-mail address is aiello@molphys.leidenuniv.nl.

References
5. Very recently, the same dispersion relation we first introduced in Ref. 3 has been used by G. F. Calvo, A. Picón, and E. Bagan, arXiv.org e-print archive, http://arxiv.org/quant-ph/0509040, to illustrate some interesting properties of photon angular momentum.
8. See Ref. 3 for a full discussion about the physical meaning of the parameter \(\theta_i\).